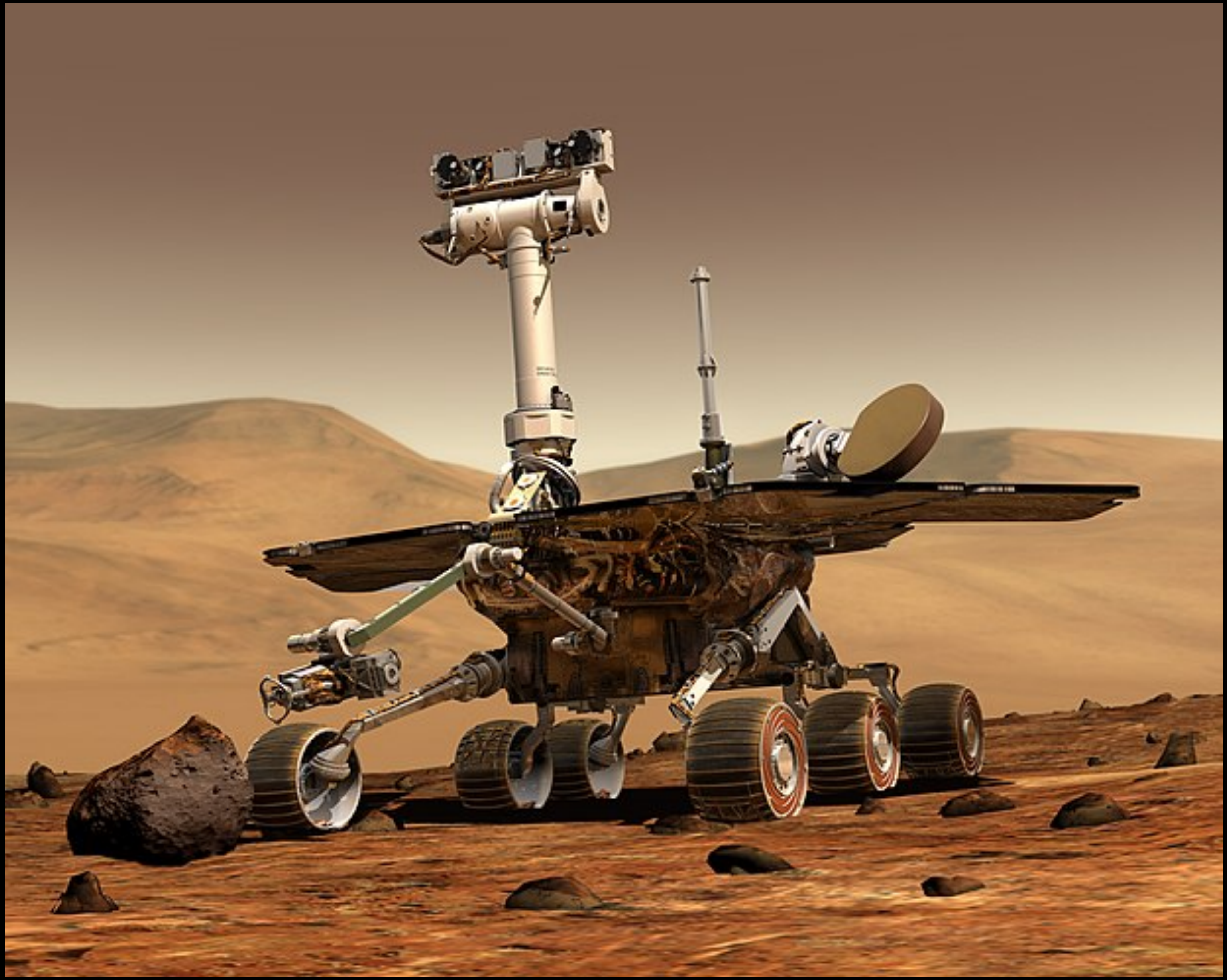
Introduction to

Artificial Intelligence

with Python

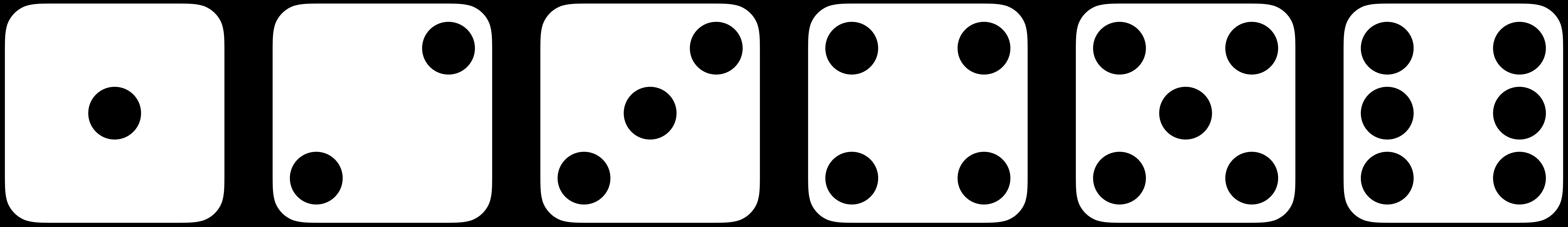
Uncertainty



NEXT 36 HOURS

HOURLY → | 10 DAYS →

TONIGHT	THU	THU NIGHT	FRI	FRI NIGHT
CLEAR				
LOW	HIGH	LOW	HIGH	LOW
20°	36°	25°	46°	32°
0%	0%	0%	0%	20%



Probability

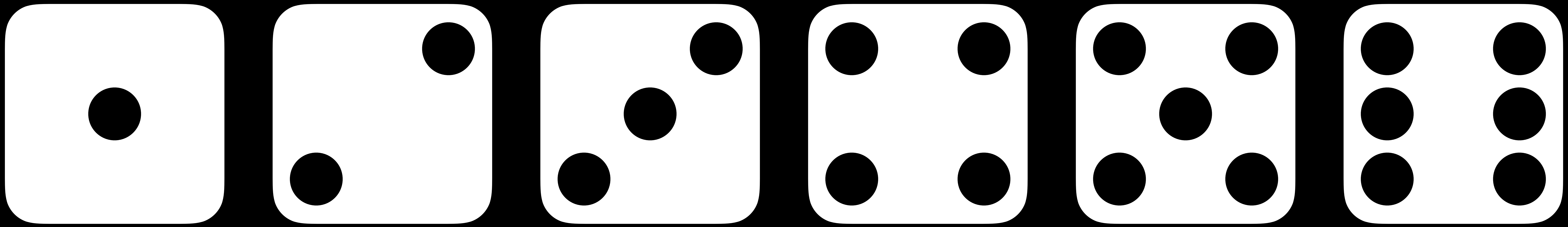
Possible Worlds

$$\omega$$

$$P(\omega)$$

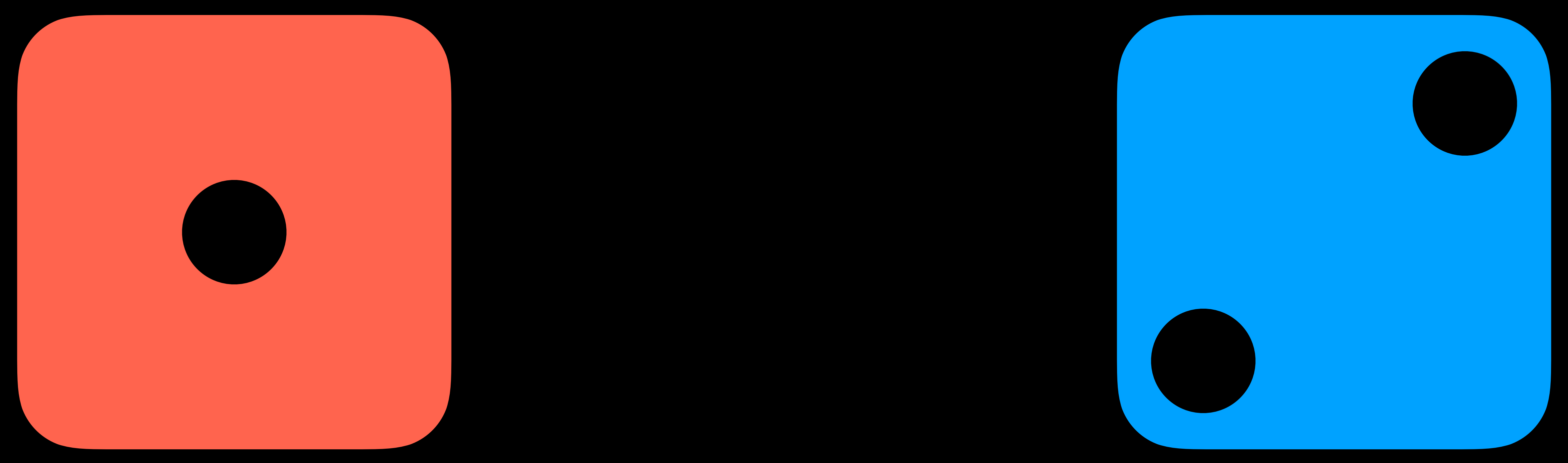
$$0 \leq P(\omega) \leq 1$$

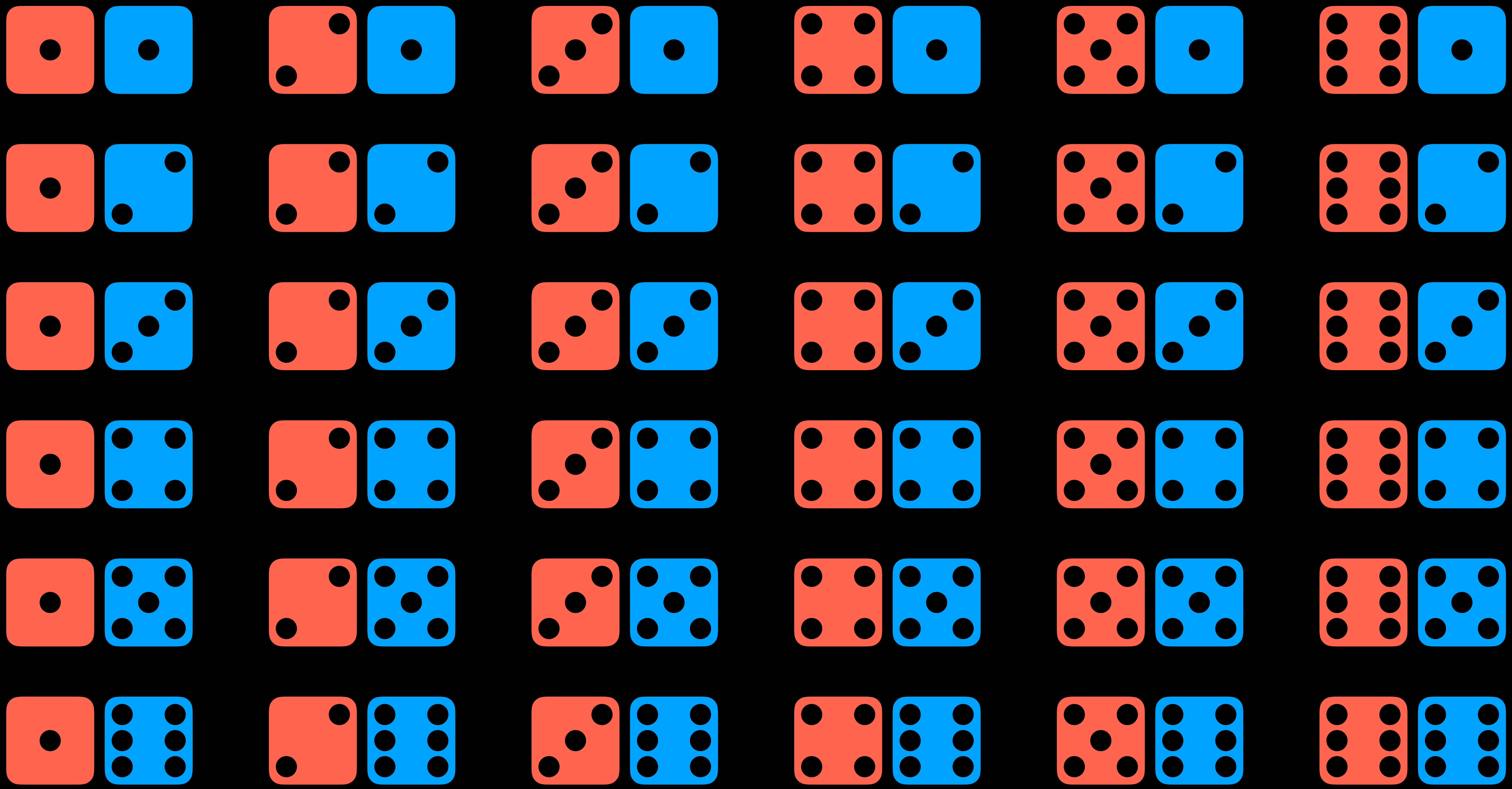
$$\sum_{\omega \in \Omega} P(\omega) = 1$$



$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

$$P(\text{⚁}) = \frac{1}{6}$$





2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

$$P(sum\ to\ 12) = \frac{1}{36}$$

$$P(sum\ to\ 7) = \frac{6}{36} = \frac{1}{6}$$

unconditional probability

degree of belief in a proposition
in the absence of any other evidence

conditional probability

degree of belief in a proposition given some evidence that has already been revealed

conditional probability

$$P(a \mid b)$$

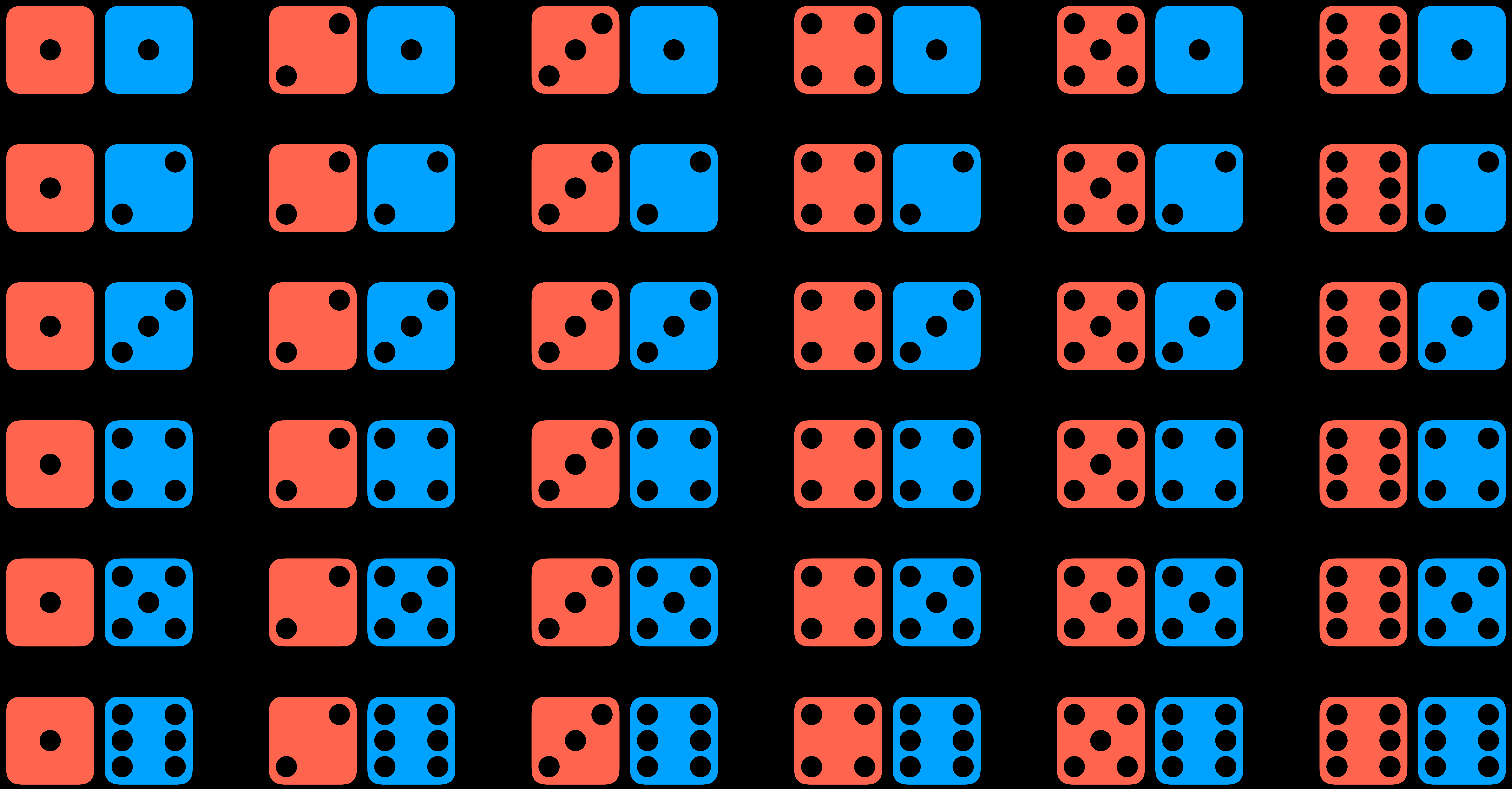
$$P(rain\ today \mid rain\ yesterday)$$

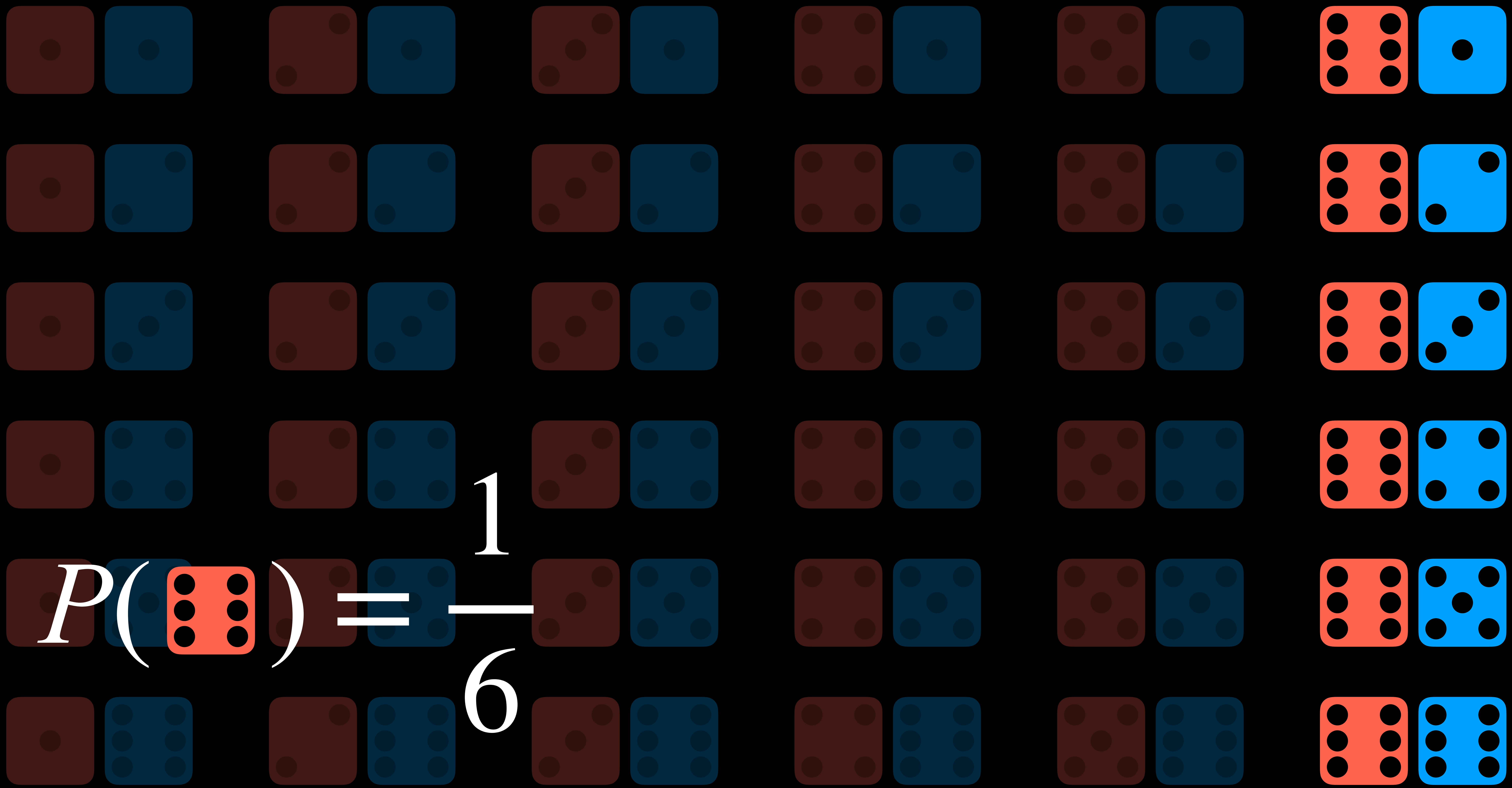
$$P(route\ change \mid traffic\ conditions)$$

$$P(disease \mid test\ results)$$

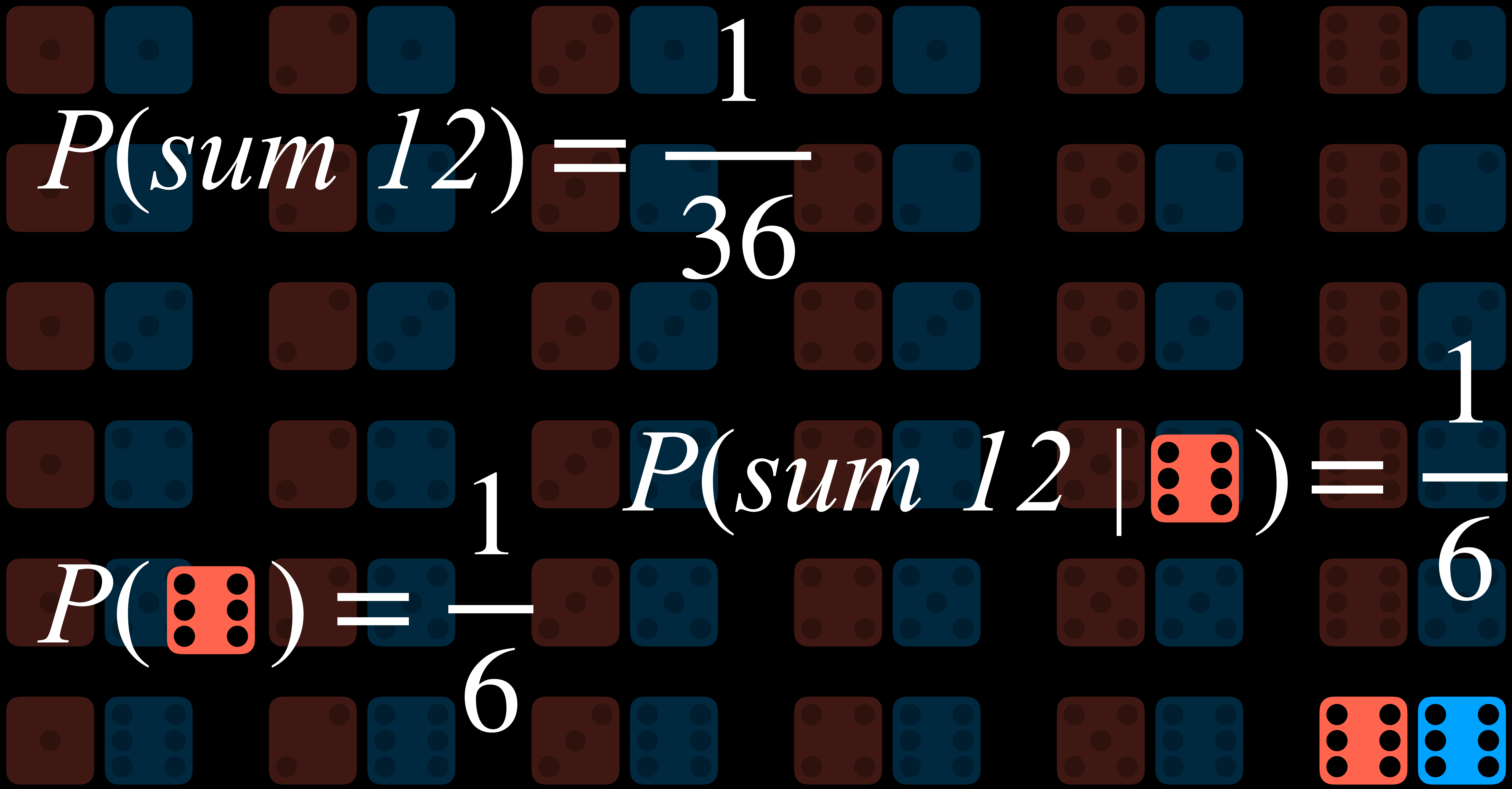
$$P(a \mid b) = \frac{P(a \wedge b)}{P(b)}$$

$$P(sum\ 12 \mid \text{⚅})$$





$$P(\text{⚅}) = \frac{1}{6}$$



$$P(\text{sum } 12) = \frac{1}{36}$$

$$P(\text{⚅} \mid \text{sum } 12) = \frac{1}{6}$$

$$P(\text{⚅}) = \frac{1}{6}$$

$$P(a \mid b) = \frac{P(a \land b)}{P(b)}$$

$$P(a \land b) = P(b)P(a \mid b)$$

$$P(a \land b) = P(a)P(b \mid a)$$

random variable

a variable in probability theory with a domain of possible values it can take on random variable

$Roll$

$\{1, 2, 3, 4, 5, 6\}$

random variable

$Weather$

$\{sun, cloud, rain, wind, snow\}$

random variable

$$Traffic$$

$$\{none, light, heavy\}$$

random variable

$$Flight$$

$$\{on\ time,\ delayed,\ cancelled\}$$

probability distribution

$$P(\mathit{Flight} = \mathit{on\ time}) = 0.6$$
$$P(\mathit{Flight} = \mathit{delayed}) = 0.3$$
$$P(\mathit{Flight} = \mathit{cancelled}) = 0.1$$

probability distribution

$$\mathbf{P}(\mathit{Flight}) = \langle 0.6, 0.3, 0.1 \rangle$$

independence

the knowledge that one event occurs does not affect the probability of the other event

independence

$$P(a \wedge b) = P(a)P(b \mid a)$$

independence

$$P(a \land b) = P(a)P(b)$$

independence

$P($ $) = P($$)P($$)$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

independence

$$P(\text{⚅⚃}) \neq P(\text{⚅})P(\text{⚃})$$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

independence

$$P(\text{⚅⚃}) \neq P(\text{⚅})P(\text{⚃} \mid \text{⚅})$$

$$= \frac{1}{6} \cdot 0 = 0$$

Bayes' Rule

$$P(a \wedge b) = P(b)\, P(a \mid b)$$

$$P(a \wedge b) = P(a)\, P(b \mid a)$$

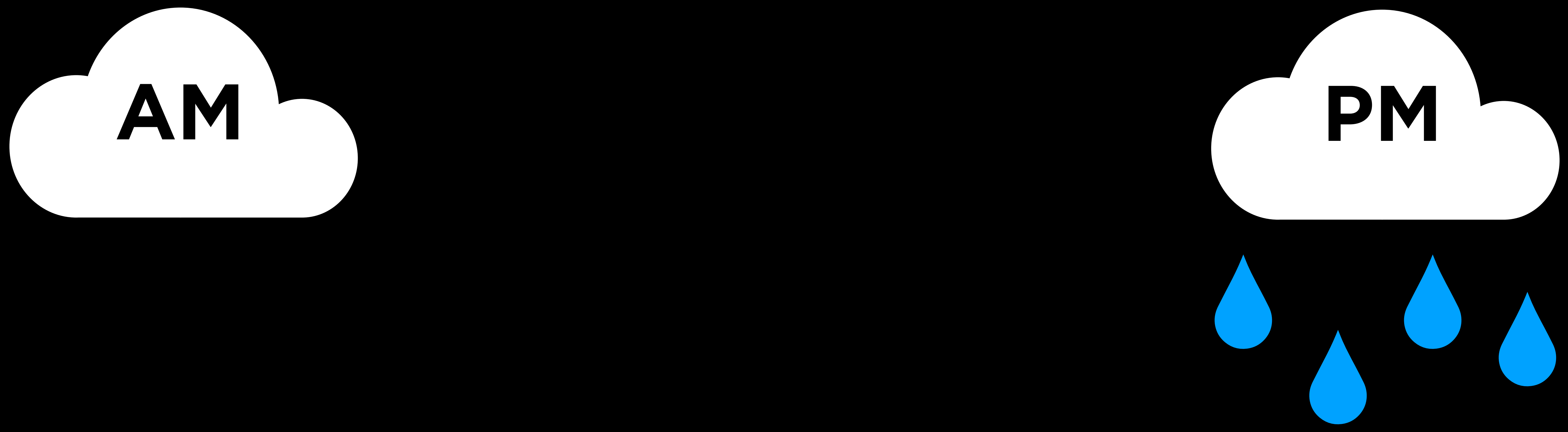
$$P(a)\,P(b \mid a) = P(b)\,P(a \mid b)$$

Bayes' Rule

$$P(b \mid a) = \frac{P(b)\ P(a \mid b)}{P(a)}$$

Bayes' Rule

$$P(b \mid a) = \frac{P(a \mid b)\, P(b)}{P(a)}$$



Given clouds in the morning,
what's the probability of rain in the afternoon?

- 80% of rainy afternoons start with cloudy mornings.
- 40% of days have cloudy mornings.
- 10% of days have rainy afternoons.

$$P(rain \mid clouds) = \frac{P(clouds \mid rain)P(rain)}{P(clouds)}$$

$$= \frac{(.8)(.1)}{.4}$$

$$= 0.2$$

Knowing

$$P(cloudy\ morning \mid rainy\ afternoon)$$

we can calculate

$$P(rainy\ afternoon \mid cloudy\ morning)$$

Knowing

$$P(\textit{visible effect} \mid \textit{unknown cause})$$

we can calculate

$$P(\textit{unknown cause} \mid \textit{visible effect})$$

Knowing

$$P(\textit{medical test result} \mid \textit{disease})$$

we can calculate

$$P(\textit{disease} \mid \textit{medical test result})$$

Knowing

$$P(blurry\ text \mid counterfeit\ bill)$$

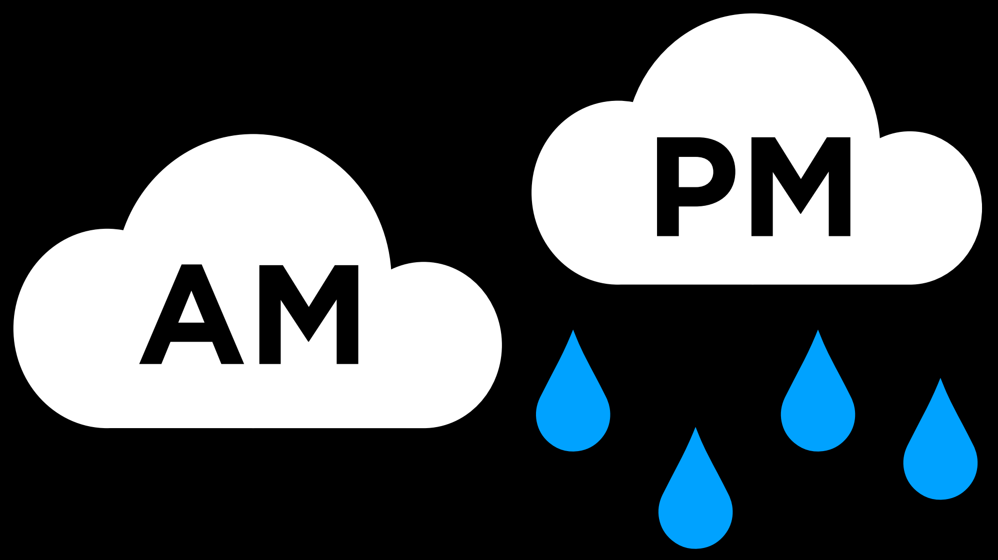
we can calculate

$$P(counterfeit\ bill \mid blurry\ text)$$

Joint Probability

$C = cloud$	$C = \neg cloud$
0.4	0.6

$R = rain$	$R = \neg rain$
0.1	0.9



	$R = rain$	$R = \neg rain$
$C = cloud$	0.08	0.32
$C = \neg cloud$	0.02	0.58

$$\mathbf{P}(C \mid rain)$$

$$\mathbf{P}(C \mid rain) = \frac{\mathbf{P}(C, rain)}{P(rain)} = \alpha\mathbf{P}(C, rain)$$

$$= \alpha\langle 0.08, 0.02\rangle = \langle 0.8, 0.2\rangle$$

	$R = rain$	$R = \neg rain$
$C = cloud$	0.08	0.32
$C = \neg cloud$	0.02	0.58

Probability Rules

Negation

$$P(\neg a) = 1 - P(a)$$

Inclusion-Exclusion

$$P(a \vee b) = P(a) + P(b) - P(a \wedge b)$$

Marginalization

$$P(a) = P(a, b) + P(a, \neg b)$$

Marginalization

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j)$$

Marginalization

	$R = rain$	$R = \neg rain$
$C = cloud$	0.08	0.32
$C = \neg cloud$	0.02	0.58

$$P(C = cloud)$$

$$= P(C = cloud, R = rain) + P(C = cloud, R = \neg rain)$$

$$= 0.08 + 0.32$$

$$= 0.40$$

Conditioning

$$P(a) = P(a \mid b)P(b) + P(a \mid \neg b)P(\neg b)$$

Conditioning

$$P(X = x_i) = \sum_j P(X = x_i \mid Y = y_j) P(Y = y_j)$$

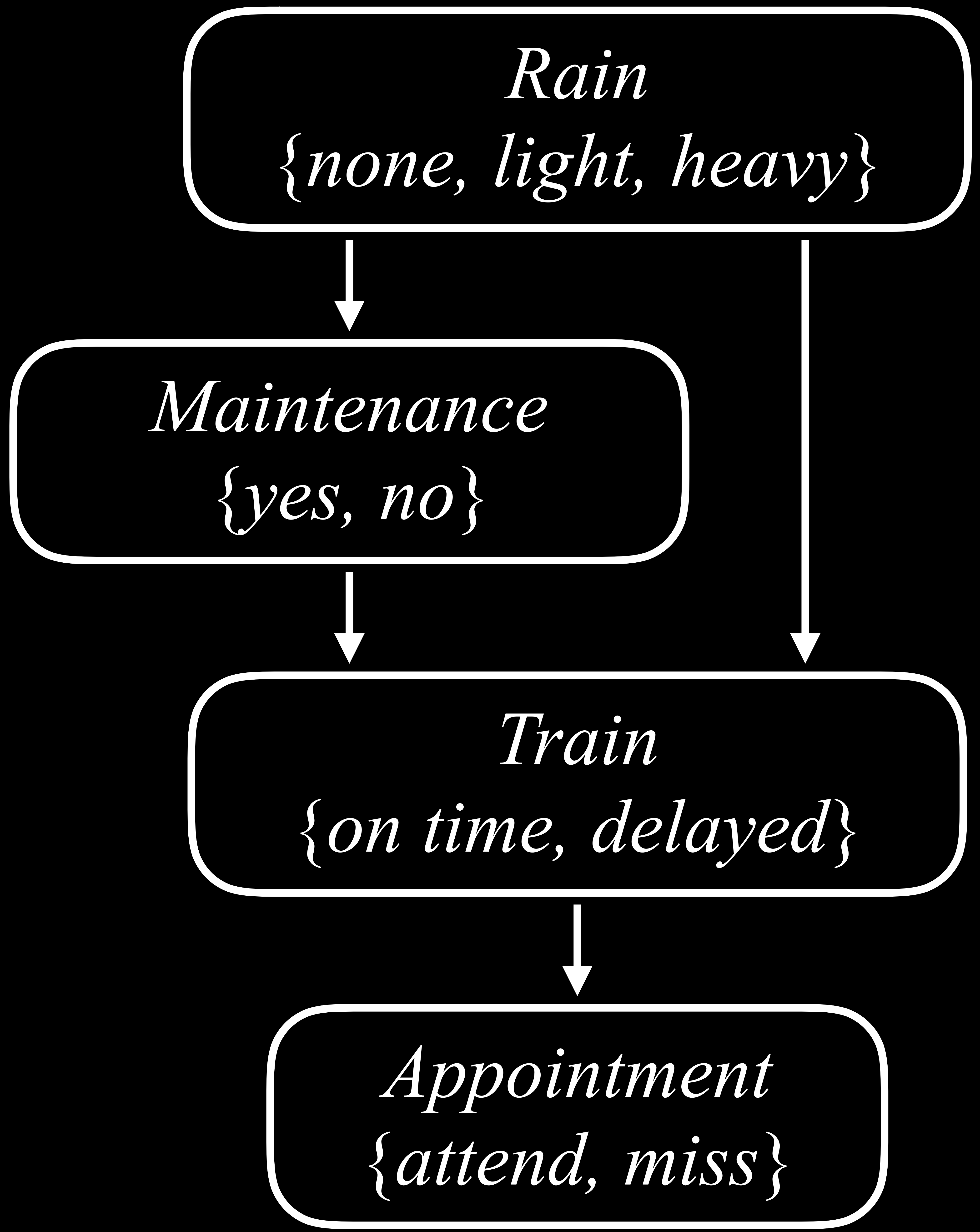
Bayesian Networks

Bayesian network

data structure that represents the dependencies among random variables

Bayesian network

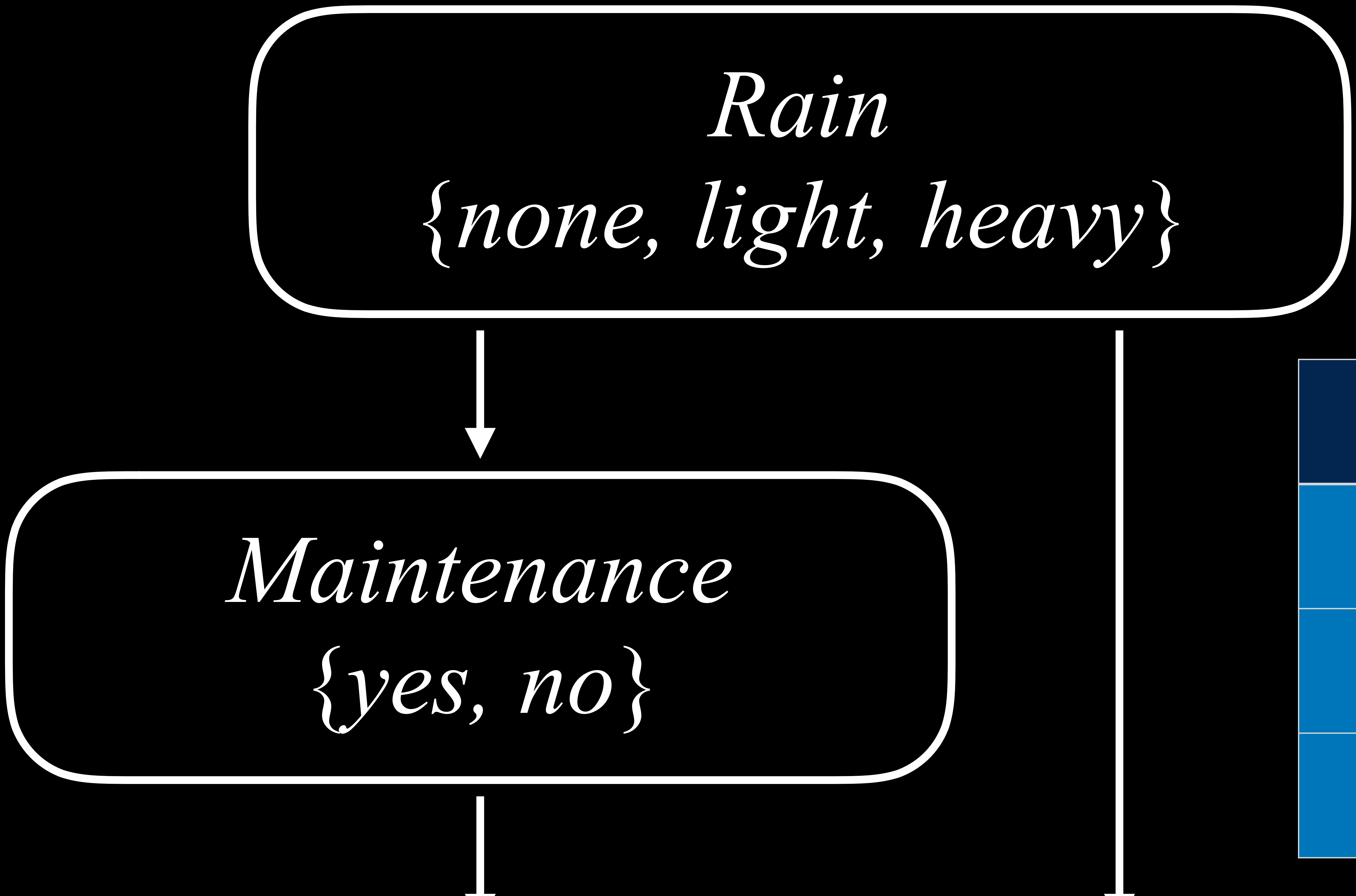
- directed graph
- each node represents a random variable
- arrow from X to Y means X is a parent of Y
- each node X has probability distribution $\mathbf{P}(X \mid Parents(X))$



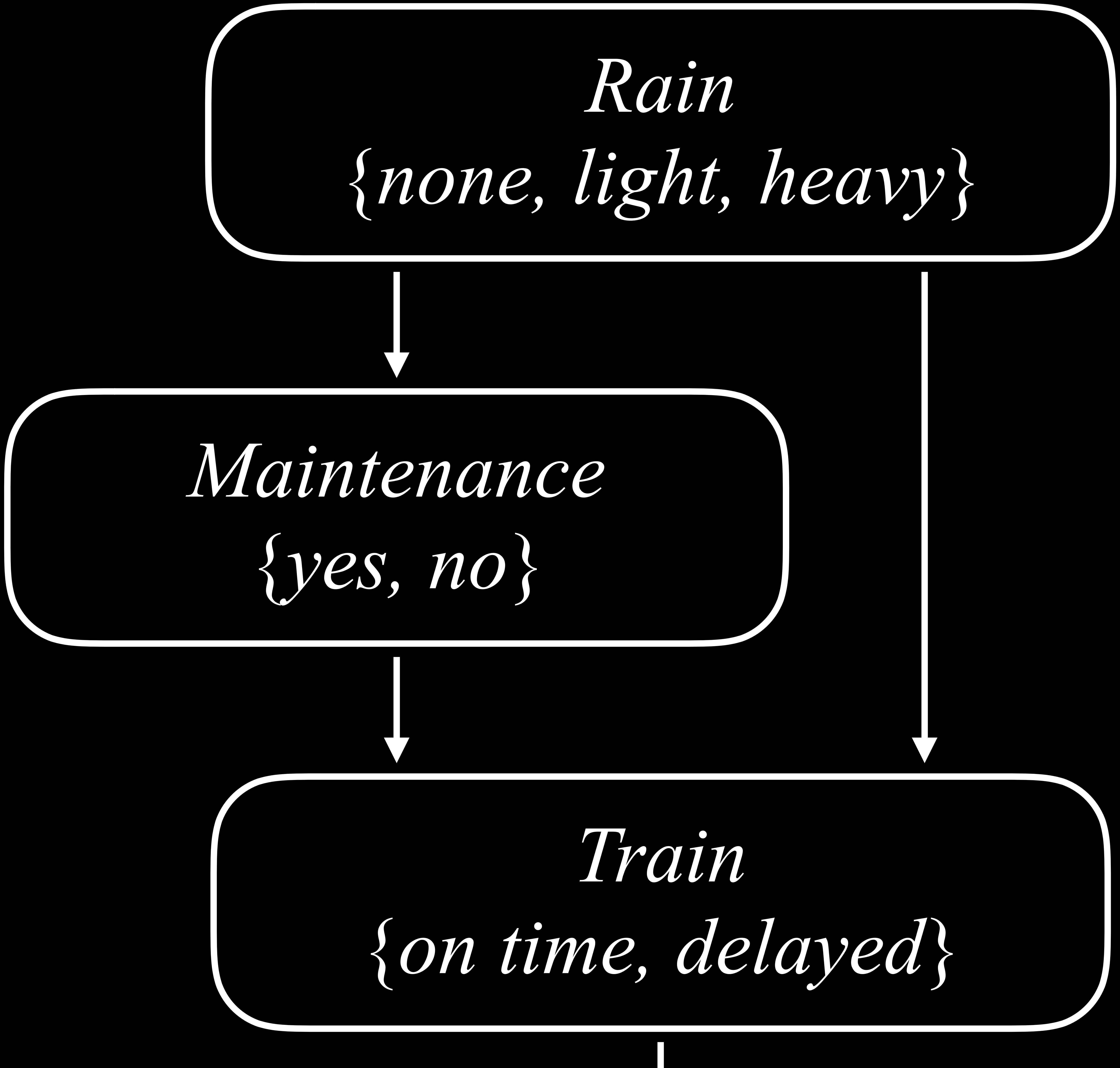
Rain
{*none, light, heavy*}
Maintenance
{*yes, no*}
Train
{*on time, delayed*}
Appointment
{*attend, miss*}

Rain
{*none*, *light*, *heavy*}

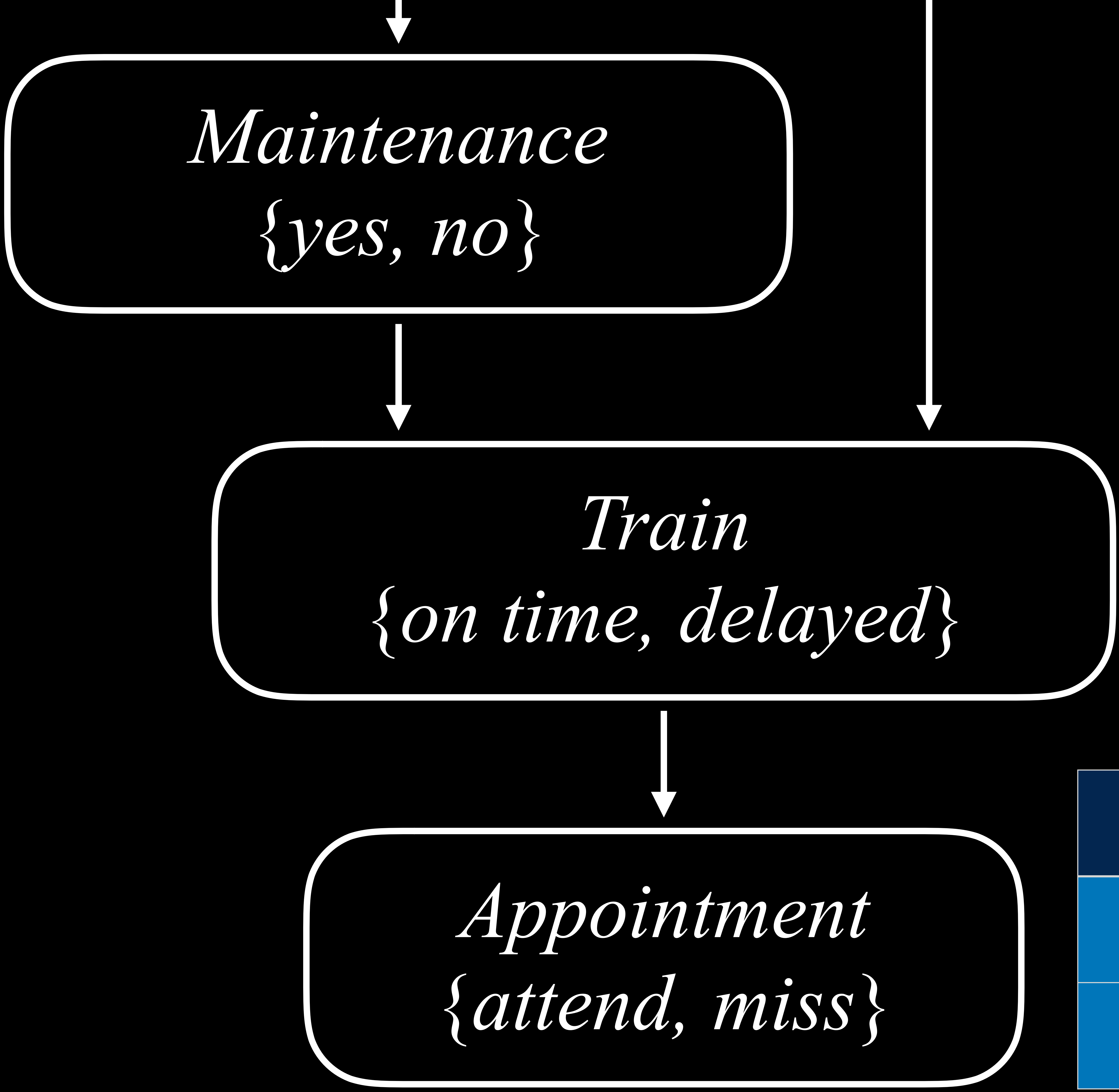
none	*light*	*heavy*
0.7	0.2	0.1



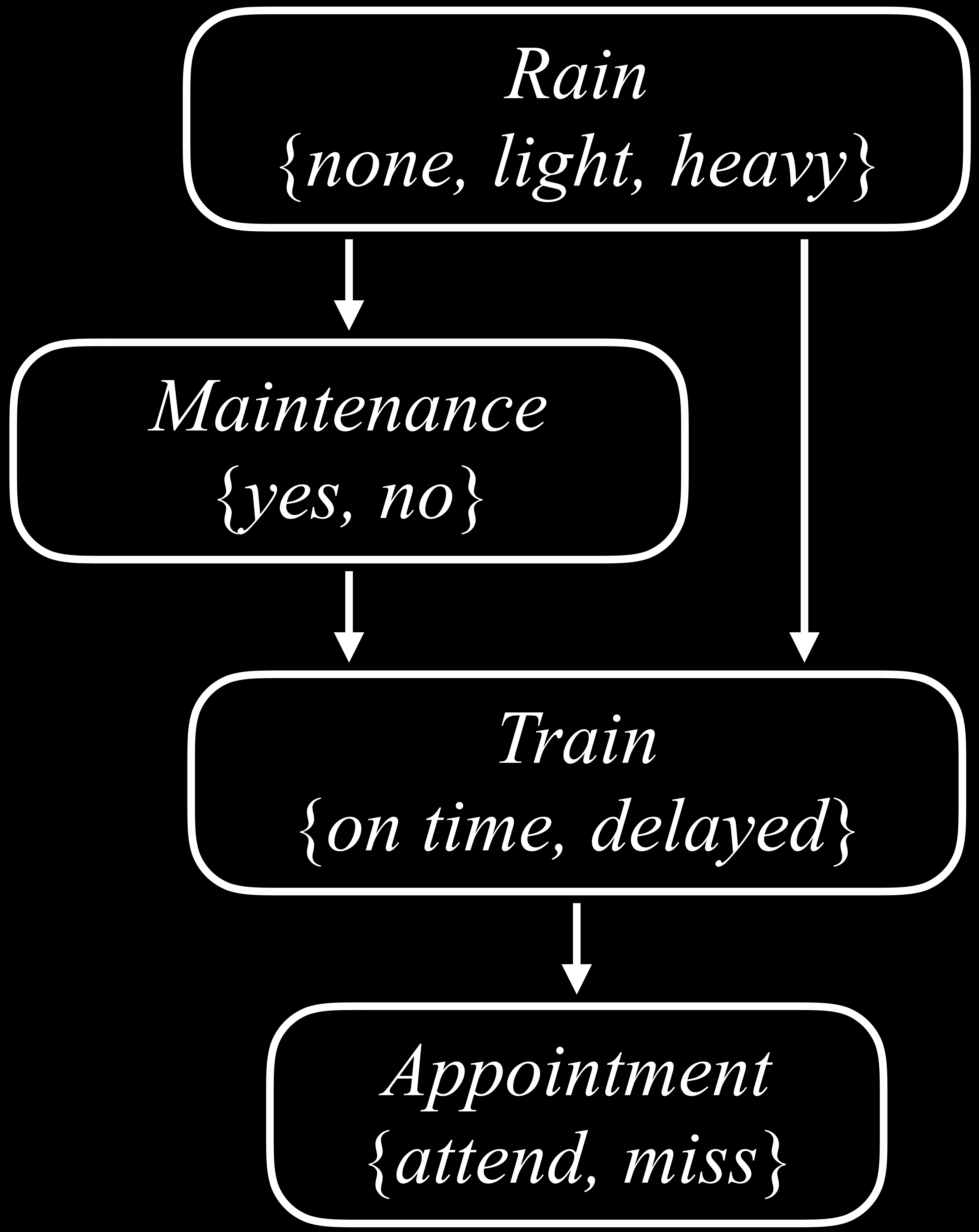
R	*yes*	*no*
none	0.4	0.6
light	0.2	0.8
heavy	0.1	0.9



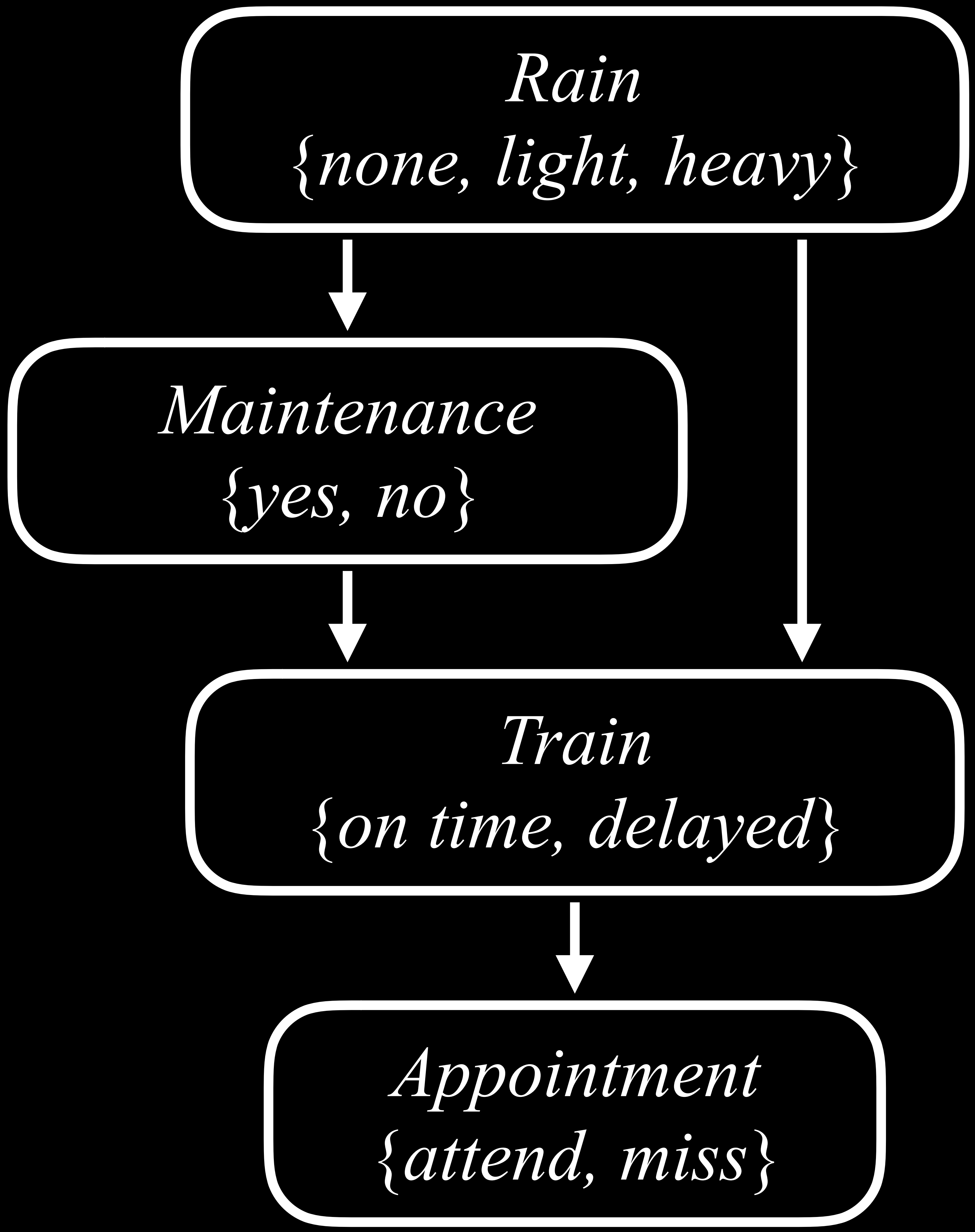
R	***M***	*on time*	*delayed*
none	*yes*	0.8	0.2
none	*no*	0.9	0.1
light	*yes*	0.6	0.4
light	*no*	0.7	0.3
heavy	*yes*	0.4	0.6
heavy	*no*	0.5	0.5



T	*attend*	*miss*
on time	0.9	0.1
delayed	0.6	0.4


Rain
{none, light, heavy}
Maintenance
{yes, no}
Train
{on time, delayed}
Appointment
{attend, miss}

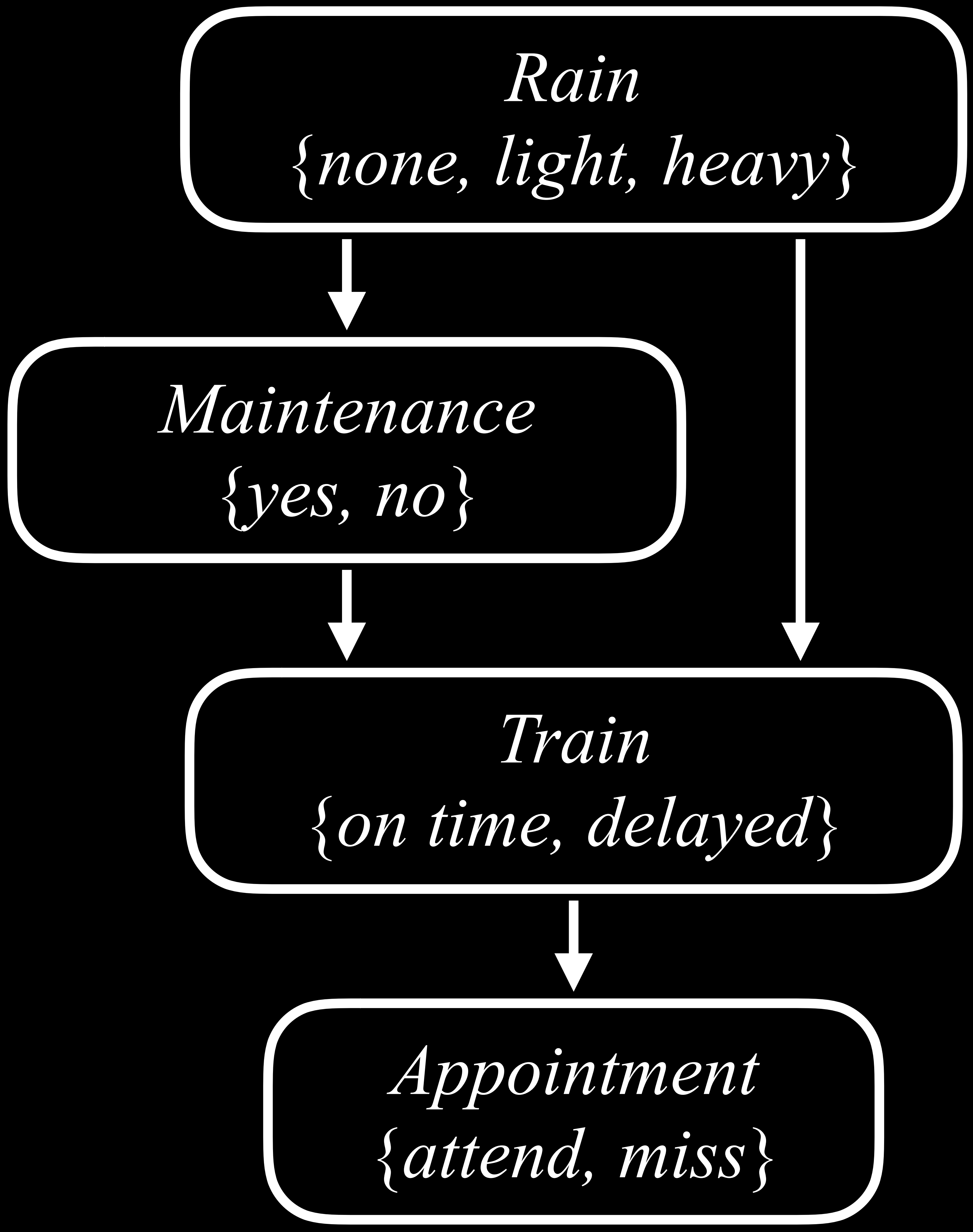
Computing Joint Probabilities



$$P(light)$$

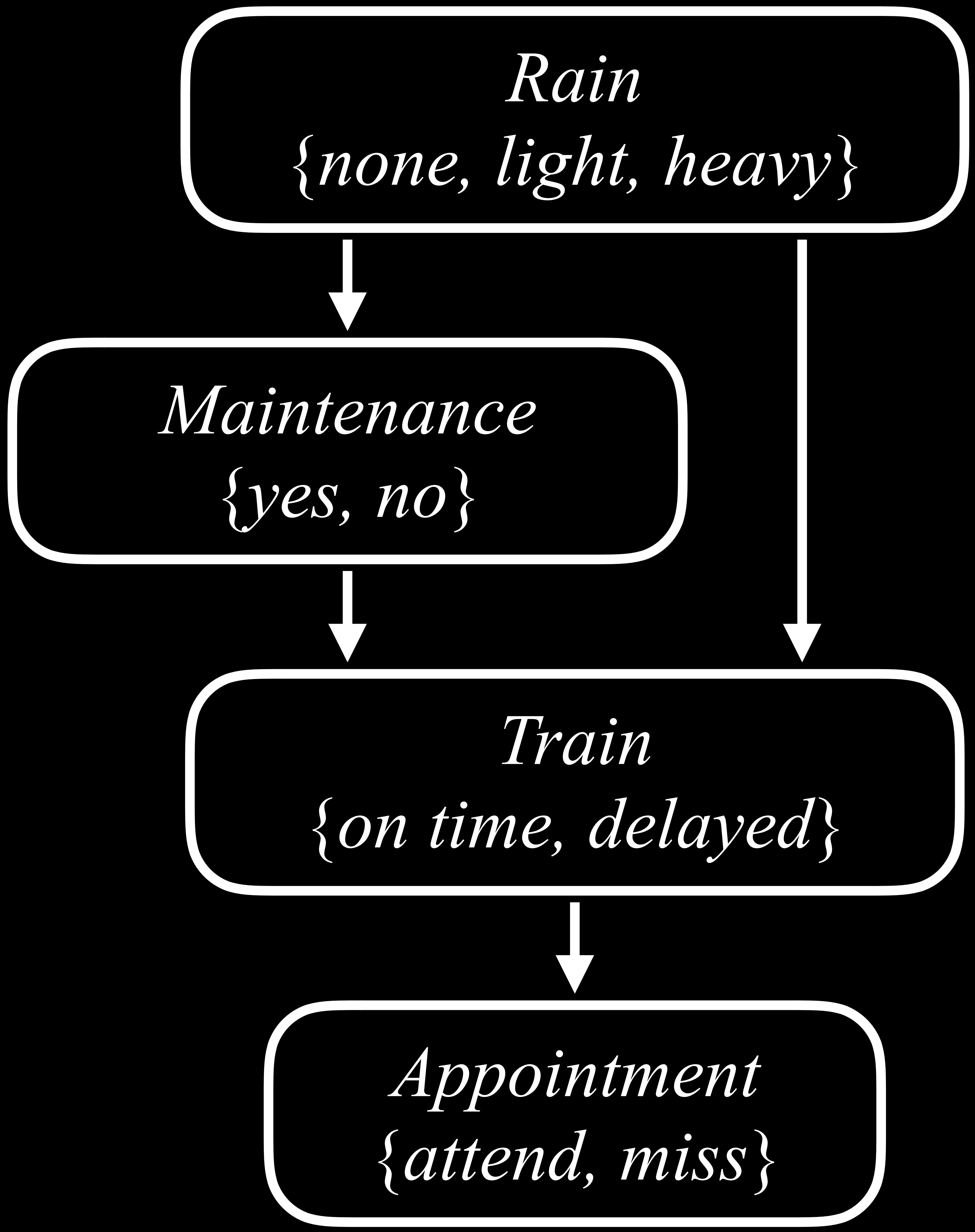
$$P(light)$$

Computing Joint Probabilities



$$P(light, no)$$

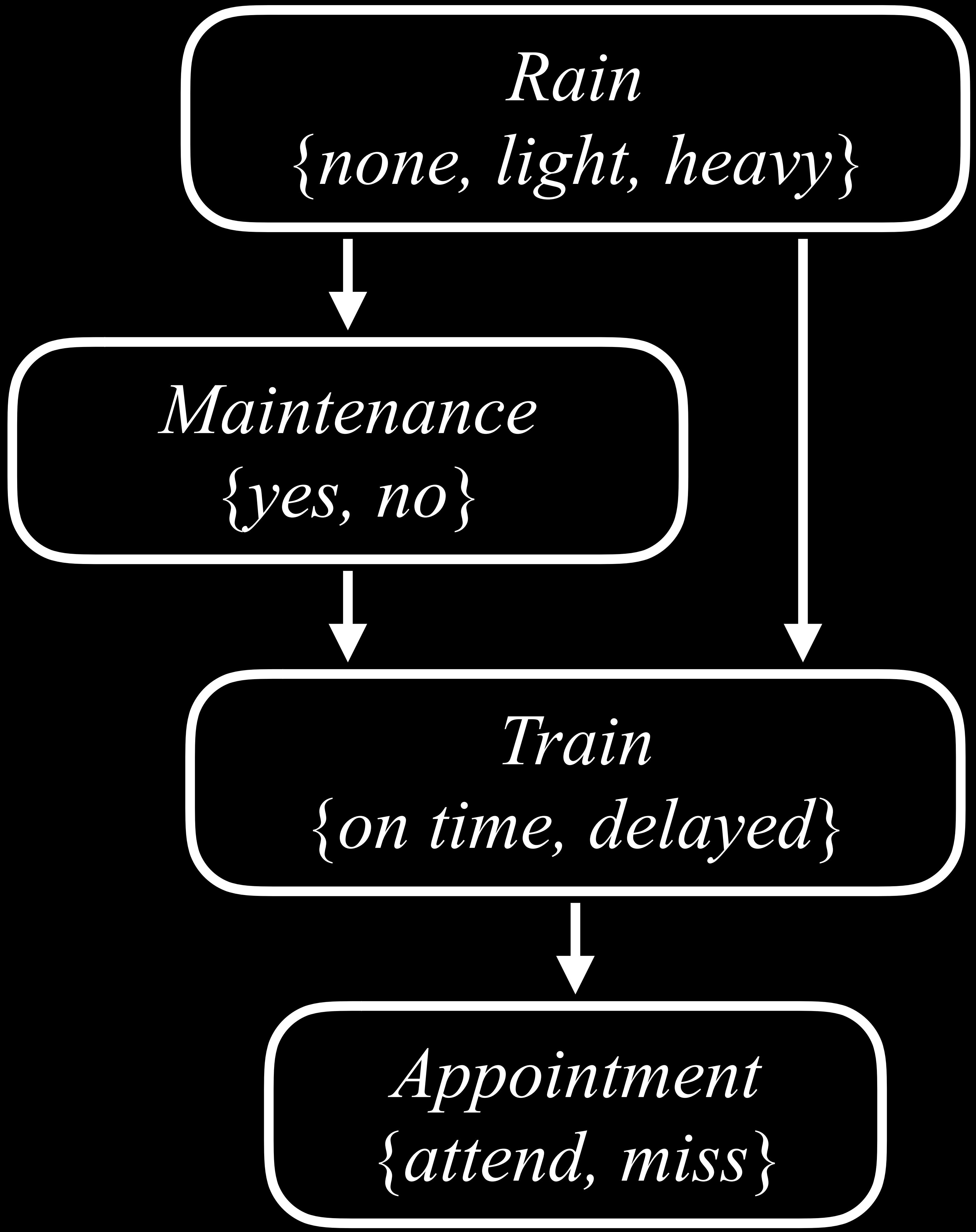
$$P(light)\ P(no \mid light)$$



Computing Joint Probabilities

$$P(light, no, delayed)$$

$$P(light)\ P(no \mid light)\ P(delayed \mid light, no)$$



Computing Joint Probabilities

$$P(light, no, delayed, miss)$$

$$P(light)\, P(no \mid light)\, P(delayed \mid light, no)\, P(miss \mid delayed)$$

Inference

Inference

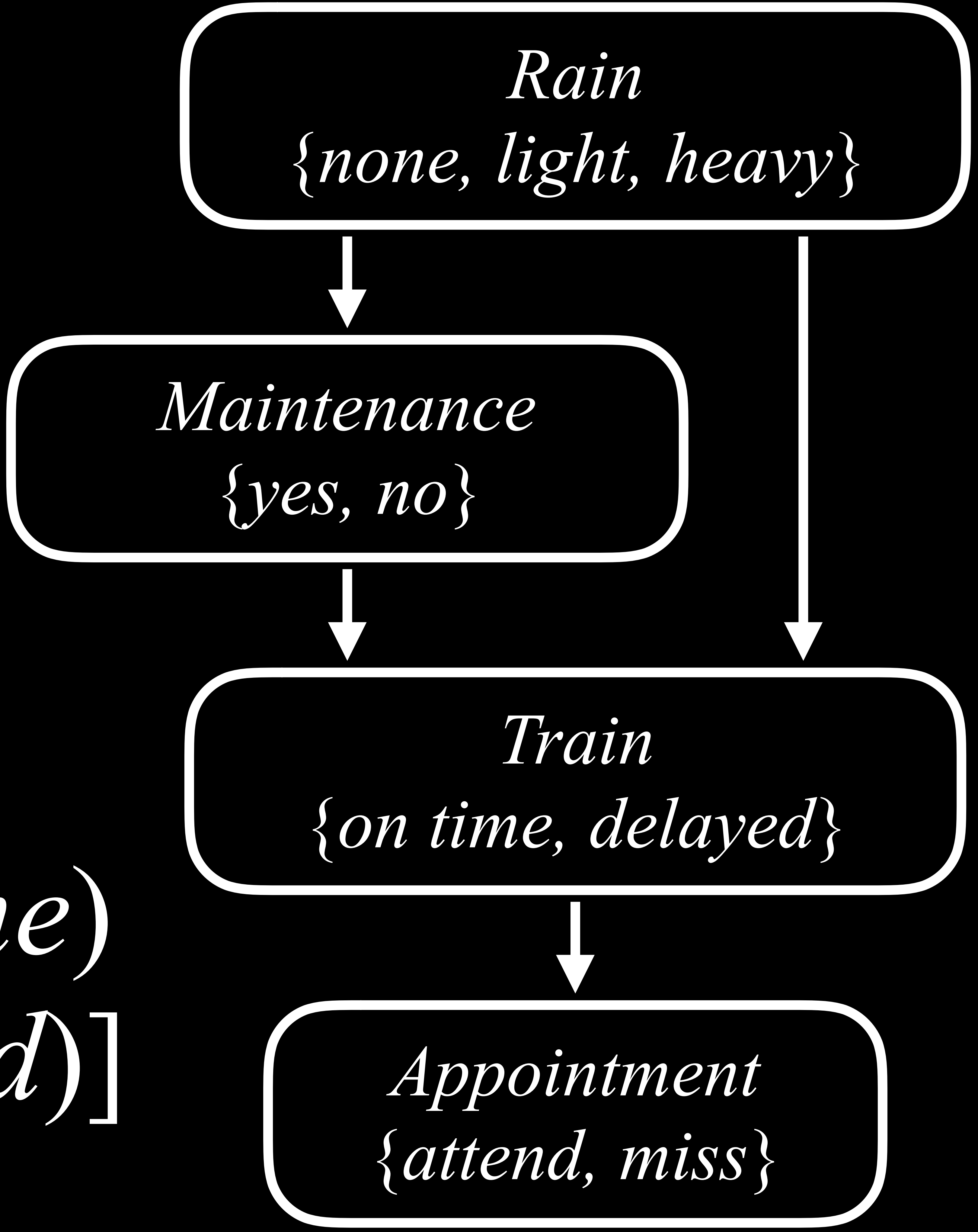
- Query $\mathbf{X}$: variable for which to compute distribution
- Evidence variables $\mathbf{E}$: observed variables for event $\mathbf{e}$
- Hidden variables $\mathbf{Y}$: non-evidence, non-query variable.

- Goal: Calculate $\mathbf{P}(\mathbf{X} \mid \mathbf{e})$

$$P(\text{Appointment} \mid \textit{light}, \textit{no})$$

$$= \alpha\, P(\text{Appointment}, \textit{light}, \textit{no})$$

$$= \alpha\, [P(\text{Appointment}, \textit{light}, \textit{no}, \textit{on time}) + P(\text{Appointment}, \textit{light}, \textit{no}, \textit{delayed})]$$



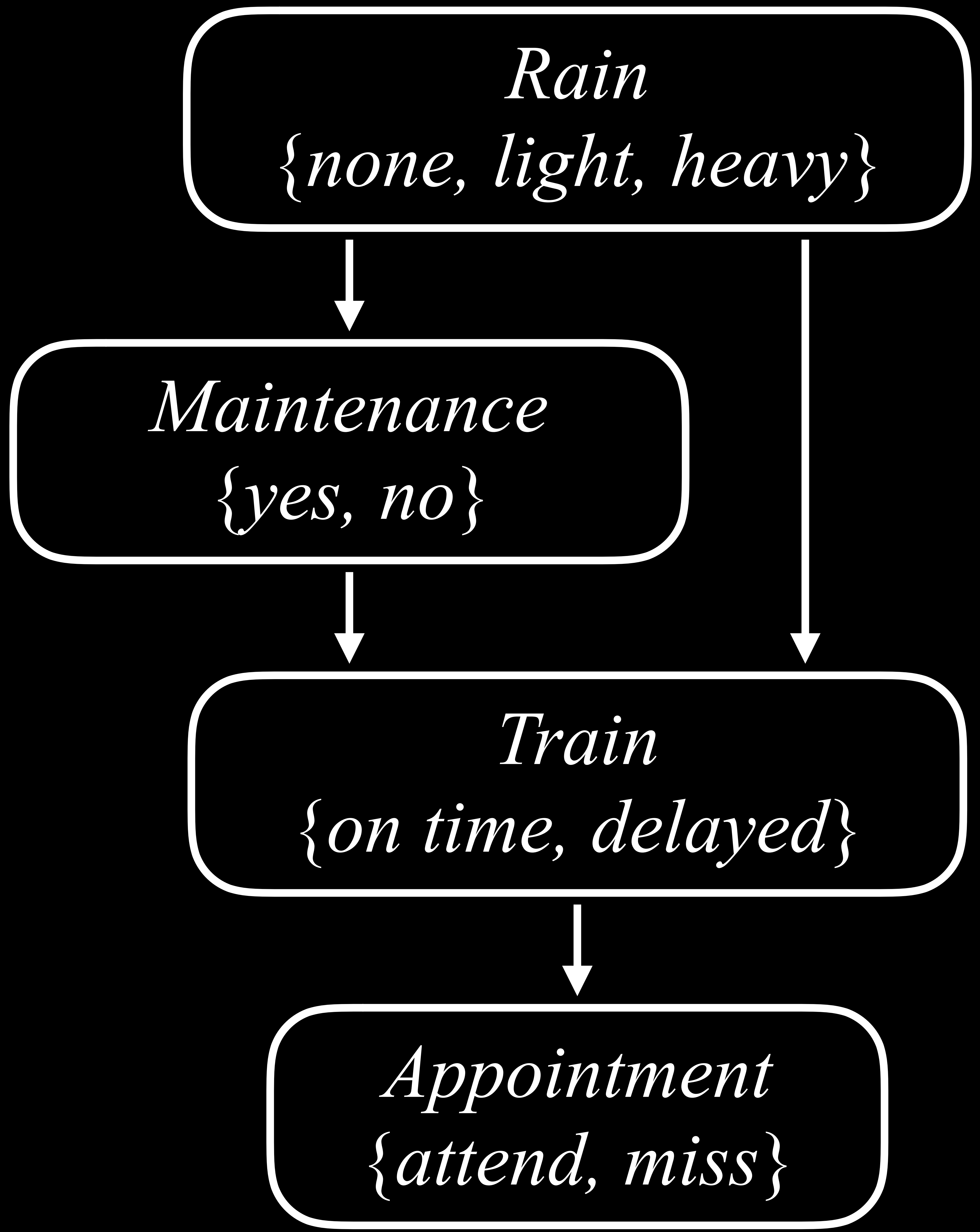
Inference by Enumeration

$$\mathbf{P}(X \mid \mathbf{e}) = \alpha\, \mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$$

X is the query variable.
$\mathbf{e}$ is the evidence.
$\mathbf{y}$ ranges over values of hidden variables.
α normalizes the result.

Approximate Inference

Sampling

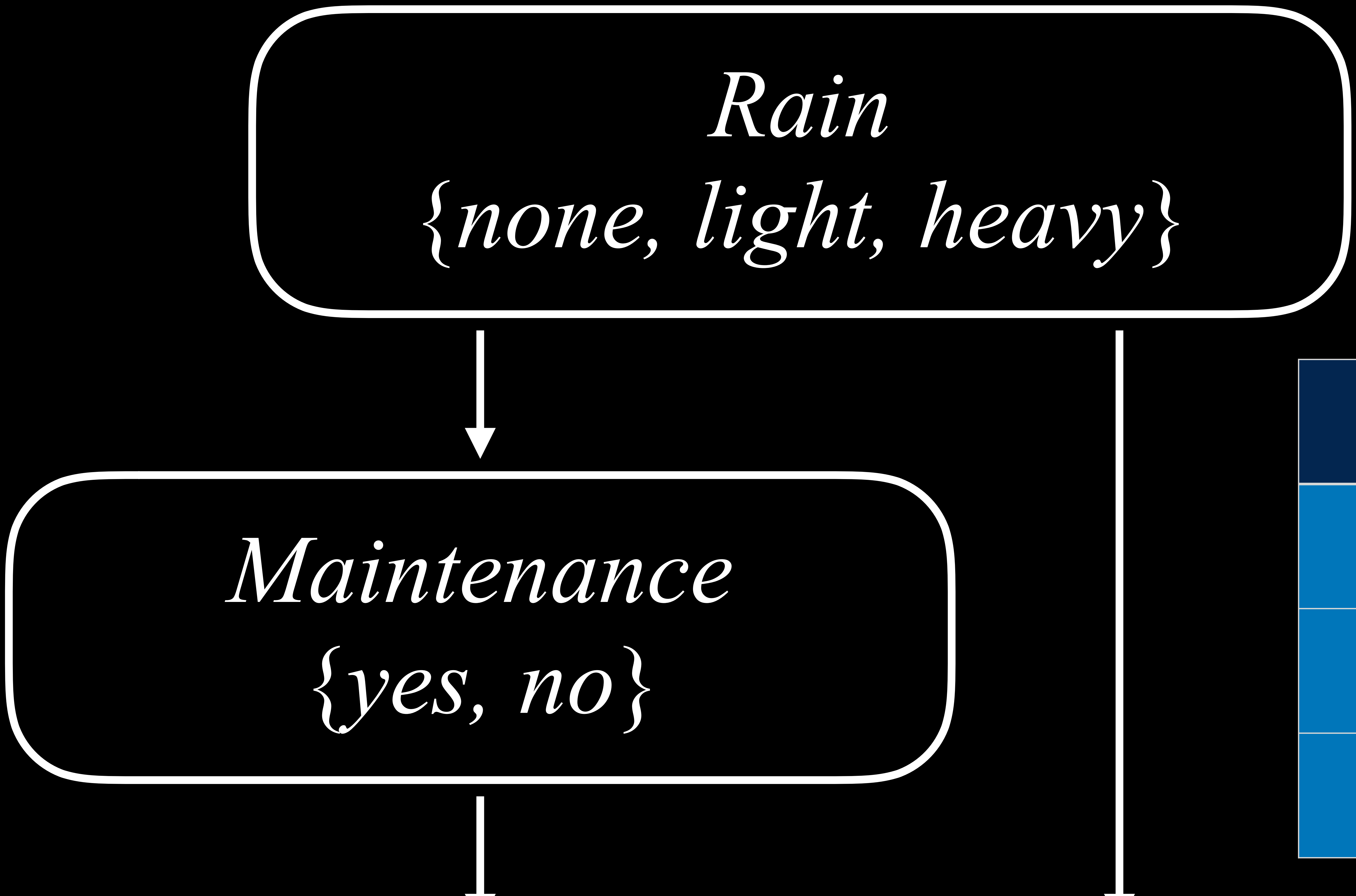

Rain
{none, light, heavy}
Maintenance
{yes, no}
Train
{on time, delayed}
Appointment
{attend, miss}

R = *none*

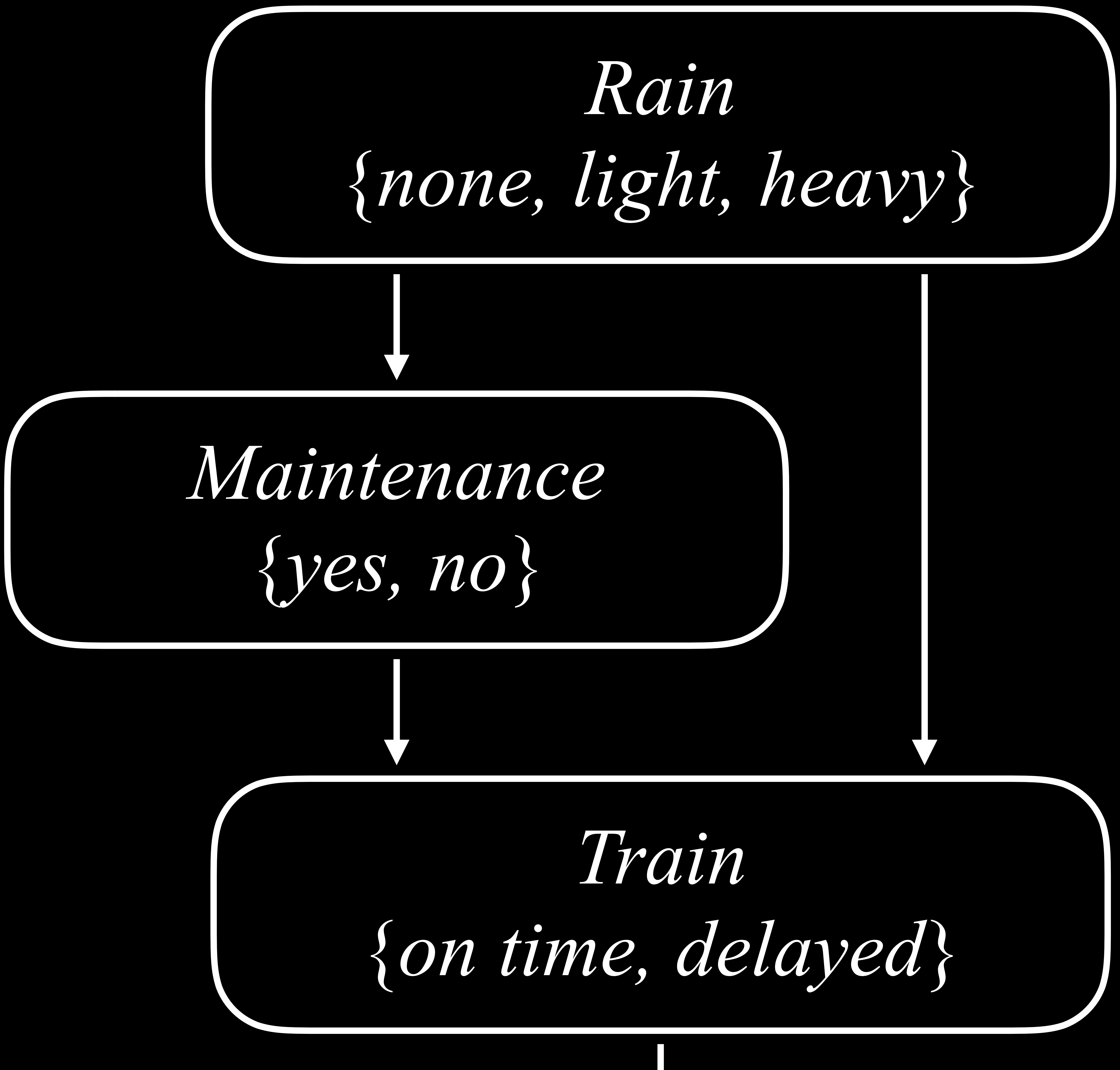
Rain
{*none*, *light*, *heavy*}

none	*light*	*heavy*
0.7	0.2	0.1

R = *none*
M = *yes*

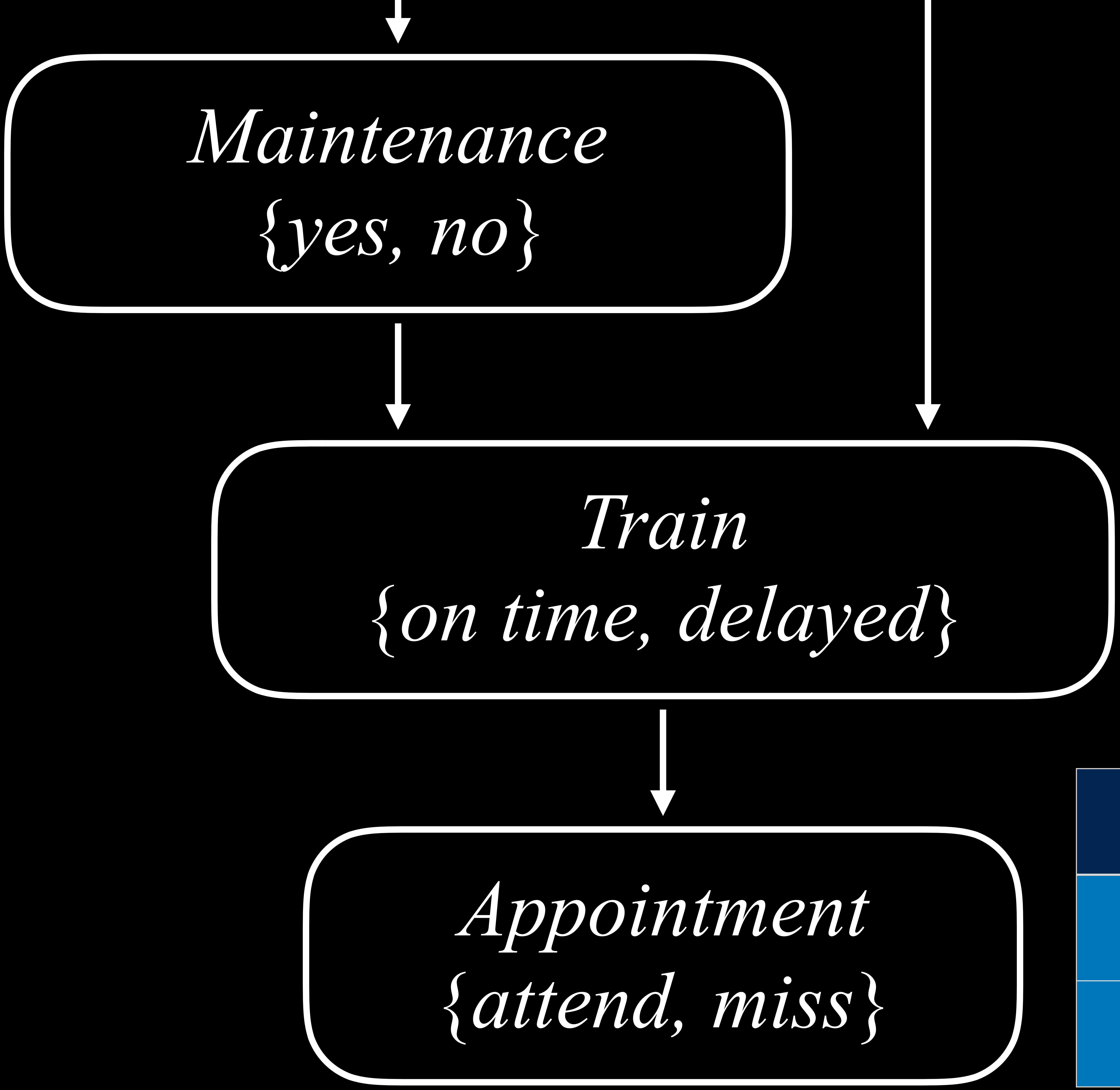


R	*yes*	*no*
none	0.4	0.6
light	0.2	0.8
heavy	0.1	0.9



R = *none*
M = *yes*
T = *on time*

R	*M*	*on time*	*delayed*
none	*yes*	0.8	0.2
none	*no*	0.9	0.1
light	*yes*	0.6	0.4
light	*no*	0.7	0.3
heavy	*yes*	0.4	0.6
heavy	*no*	0.5	0.5



R = *none*
M = *yes*
T = *on time*
A = *attend*

T	*attend*	*miss*
on time	0.9	0.1
delayed	0.6	0.4

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *light*
M = *no*
T = *on time*
A = *miss*

R = *light*
M = *yes*
T = *delayed*
A = *attend*

R = *none*
M = *no*
T = *on time*
A = *attend*

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *heavy*
M = *no*
T = *delayed*
A = *miss*

R = *light*
M = *no*
T = *on time*
A = *attend*

$$P(\mathit{Train} = \mathit{on\ time})\ ?$$

R = *light*
M = *no*
T = *on time*
A = *miss*

R = *light*
M = *yes*
T = *delayed*
A = *attend*

R = *none*
M = *no*
T = *on time*
A = *attend*

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *heavy*
M = *no*
T = *delayed*
A = *miss*

R = *light*
M = *no*
T = *on time*
A = *attend*

R = *light* M = *no* T = *on time* A = *miss*	R = *light* M = *yes* T = *delayed* A = *attend*	R = *none* M = *no* T = *on time* A = *attend*	R = *none* M = *yes* T = *on time* A = *attend*
R = *none* M = *yes* T = *on time* A = *attend*	R = *none* M = *yes* T = *on time* A = *attend*	R = *heavy* M = *no* T = *delayed* A = *miss*	R = *light* M = *no* T = *on time* A = *attend*

$$P(\text{Rain} = \textit{light} \mid \text{Train} = \textit{on time})\ ?$$

R = *light*
M = *no*
T = *on time*
A = *miss*

R = *light*
M = *yes*
T = *delayed*
A = *attend*

R = *none*
M = *no*
T = *on time*
A = *attend*

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *none*
M = *yes*
T = *on time*
A = *attend*

R = *heavy*
M = *no*
T = *delayed*
A = *miss*

R = *light*
M = *no*
T = *on time*
A = *attend*

R = *light*	R = *light*	R = *none*	R = *none*
M = *no*	M = *yes*	M = *no*	M = *yes*
T = *on time*	T = *delayed*	T = *on time*	T = *on time*
A = *miss*	A = *attend*	A = *attend*	A = *attend*

R = *none*	R = *none*	R = *heavy*	R = *light*
M = *yes*	M = *yes*	M = *no*	M = *no*
T = *on time*	T = *on time*	T = *delayed*	T = *on time*
A = *attend*	A = *attend*	A = *miss*	A = *attend*

R = *light*	R = *light*	R = *none*	R = *none*
M = *no*	M = *yes*	M = *no*	M = *yes*
T = *on time*	T = *delayed*	T = *on time*	T = *on time*
A = *miss*	A = *attend*	A = *attend*	A = *attend*

R = *none*	R = *none*	R = *heavy*	R = *light*
M = *yes*	M = *yes*	M = *no*	M = *no*
T = *on time*	T = *on time*	T = *delayed*	T = *on time*
A = *attend*	A = *attend*	A = *miss*	A = *attend*

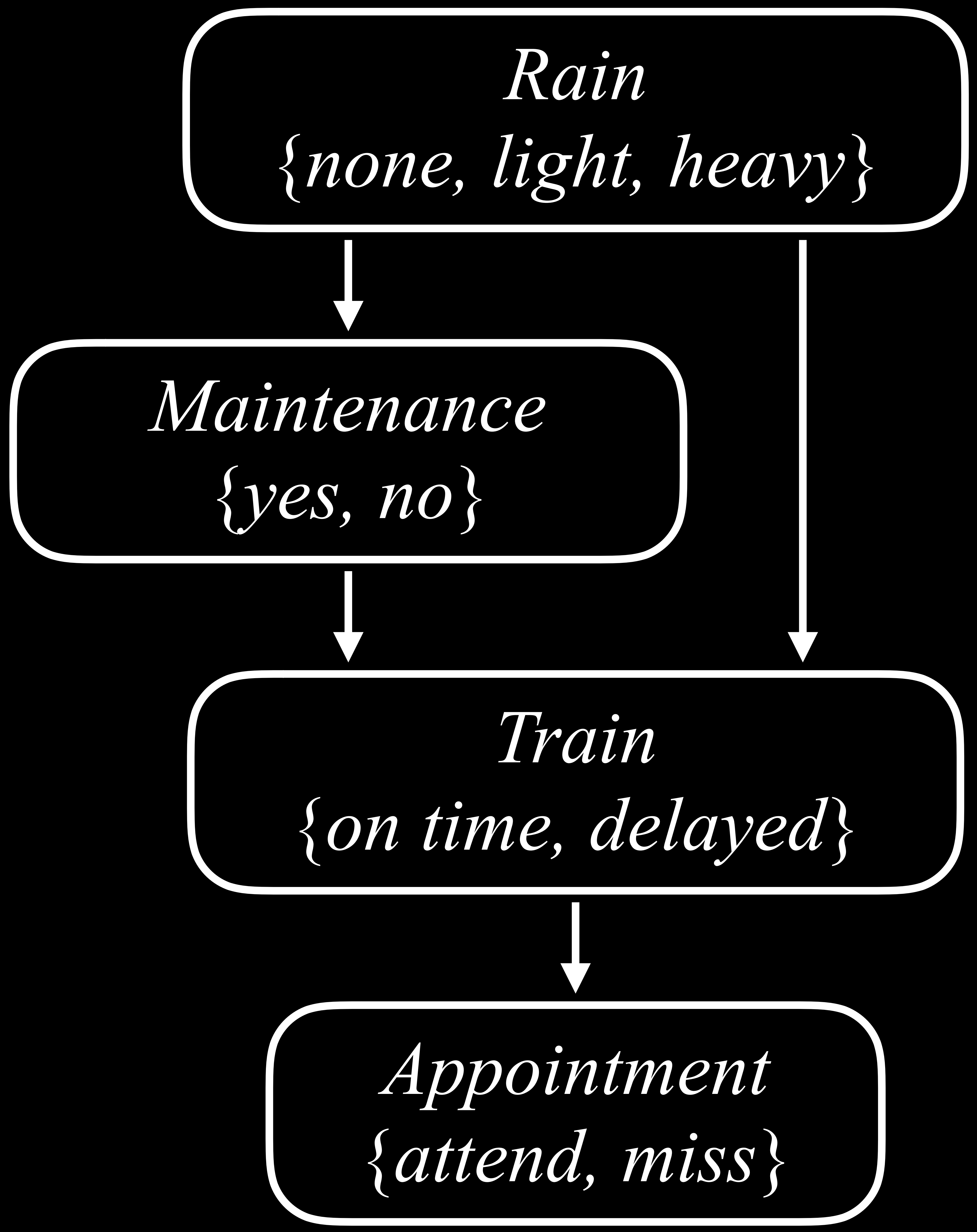
Rejection Sampling

Likelihood Weighting

Likelihood Weighting

- Start by fixing the values for evidence variables.
- Sample the non-evidence variables using conditional probabilities in the Bayesian Network.
- Weight each sample by its **likelihood**: the probability of all of the evidence.

$$P(\text{Rain} = \textit{light} \mid \text{Train} = \textit{on time})\ ?$$

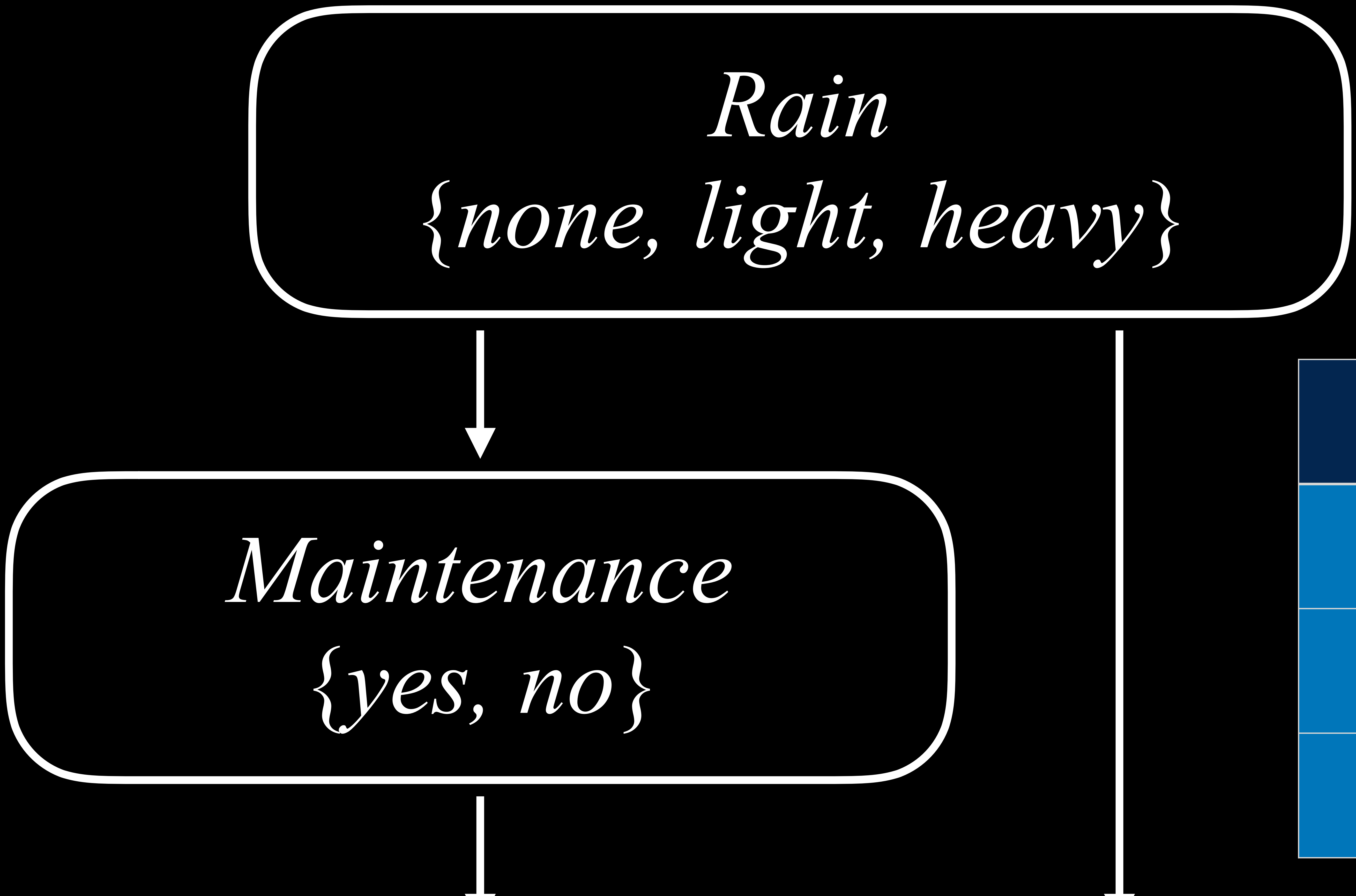

Rain
{none, light, heavy}
Maintenance
{yes, no}
Train
{on time, delayed}
Appointment
{attend, miss}

R = *light*
T = *on time*

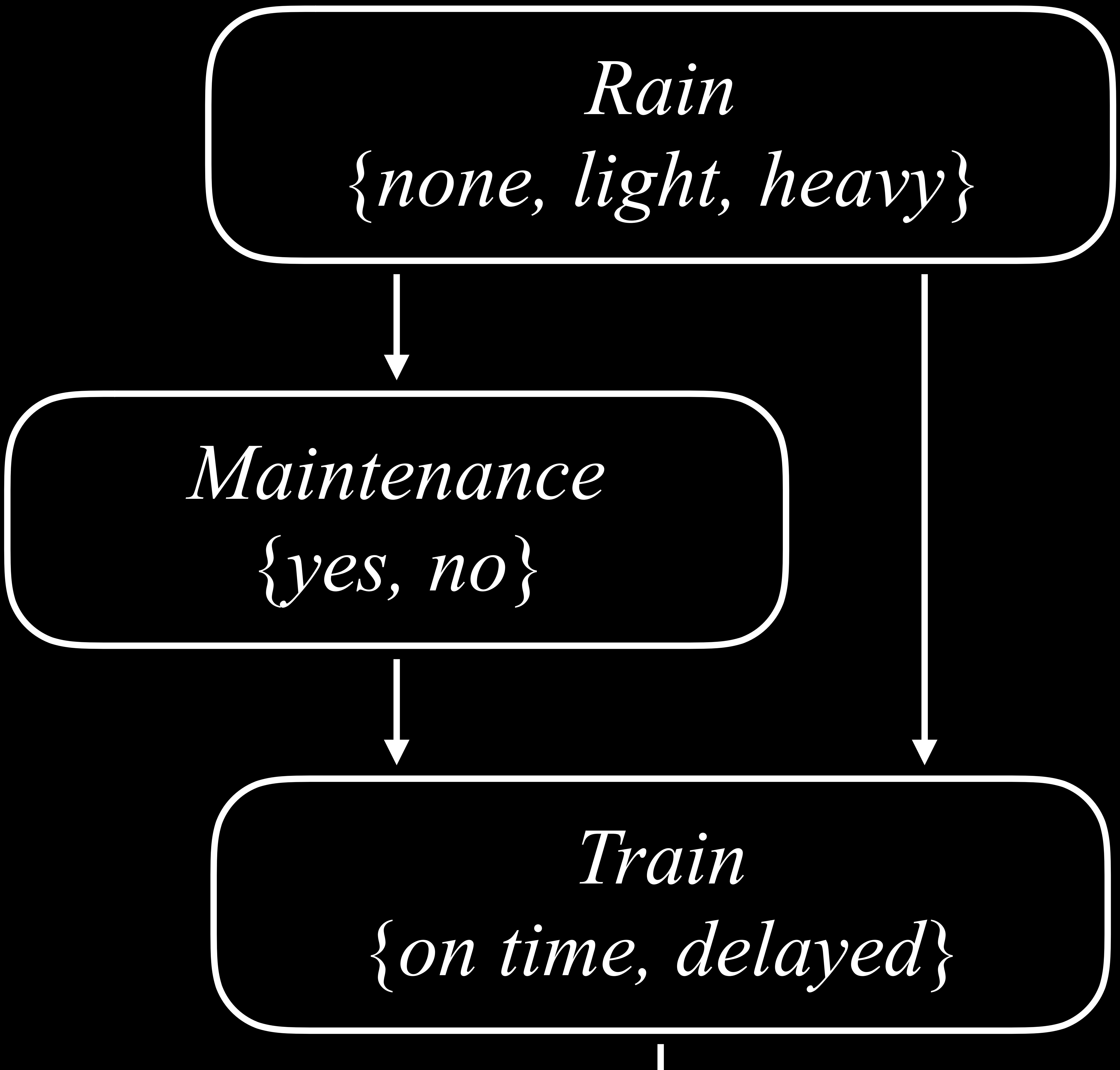
Rain
{*none*, *light*, *heavy*}

none	*light*	*heavy*
0.7	0.2	0.1

R = *light*
M = *yes*
T = *on time*

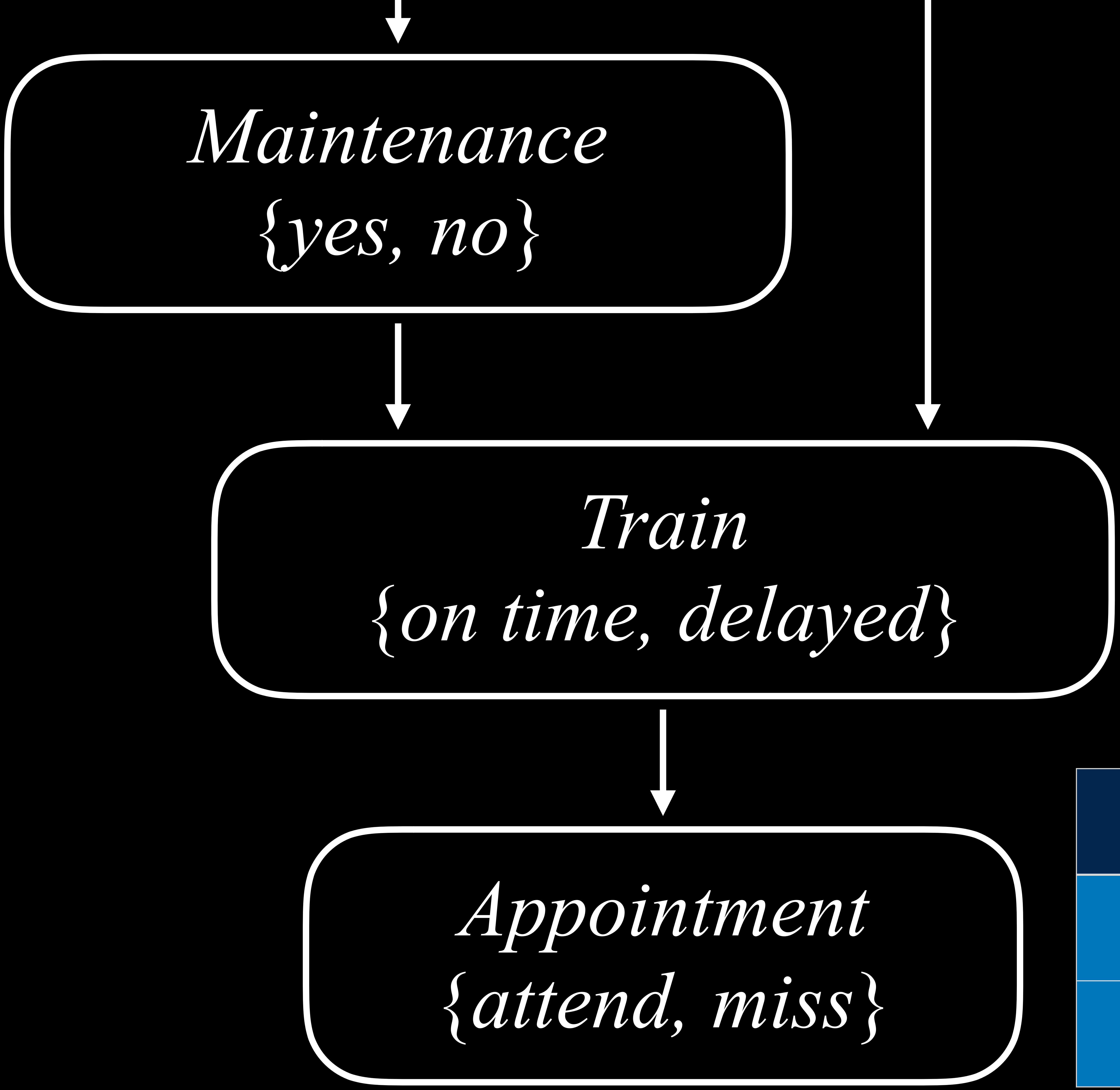


R	*yes*	*no*
none	0.4	0.6
light	0.2	0.8
heavy	0.1	0.9



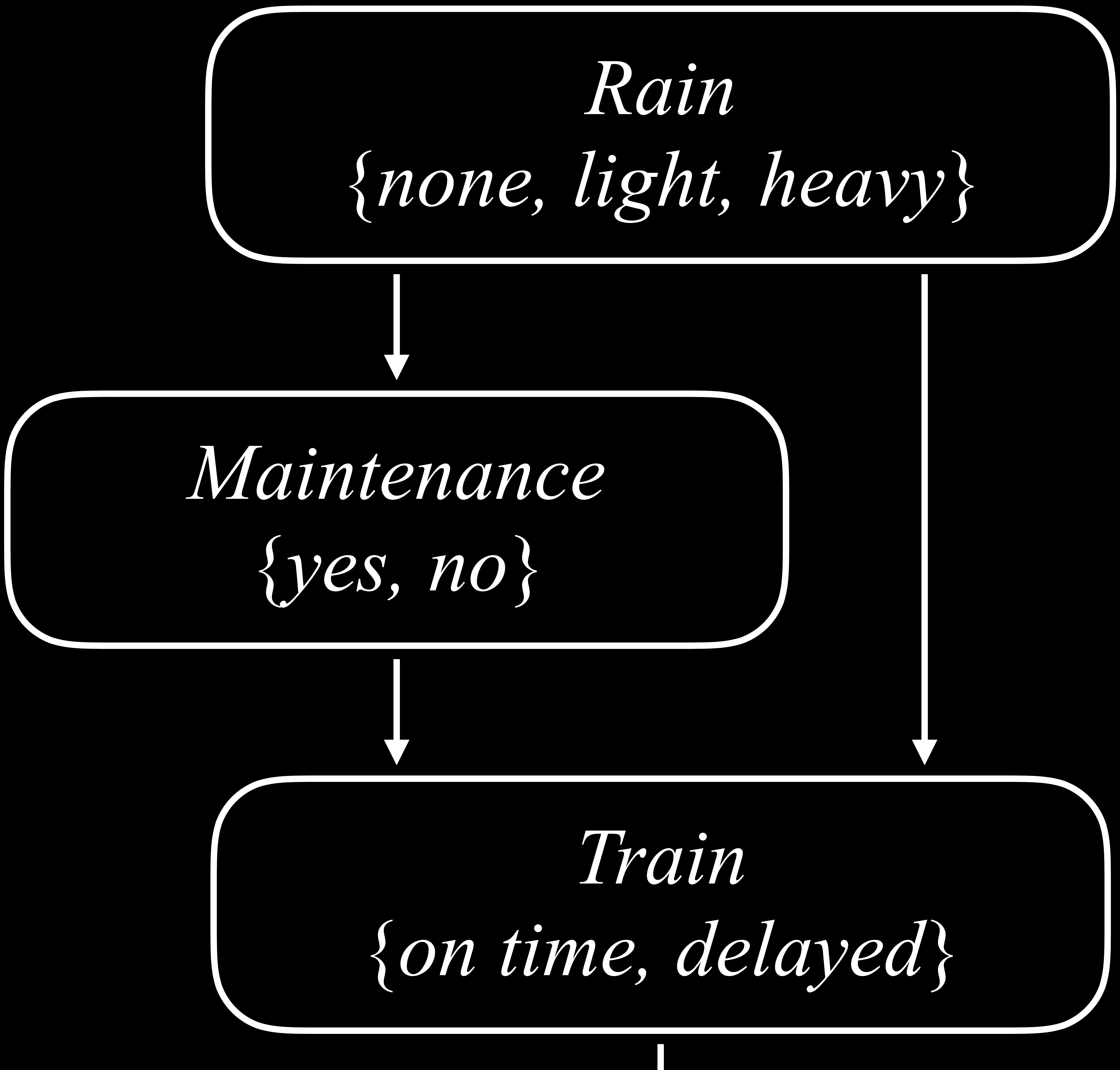
R = *light*
M = *yes*
T = *on time*

R	*M*	*on time*	*delayed*
none	*yes*	0.8	0.2
none	*no*	0.9	0.1
light	*yes*	0.6	0.4
light	*no*	0.7	0.3
heavy	*yes*	0.4	0.6
heavy	*no*	0.5	0.5



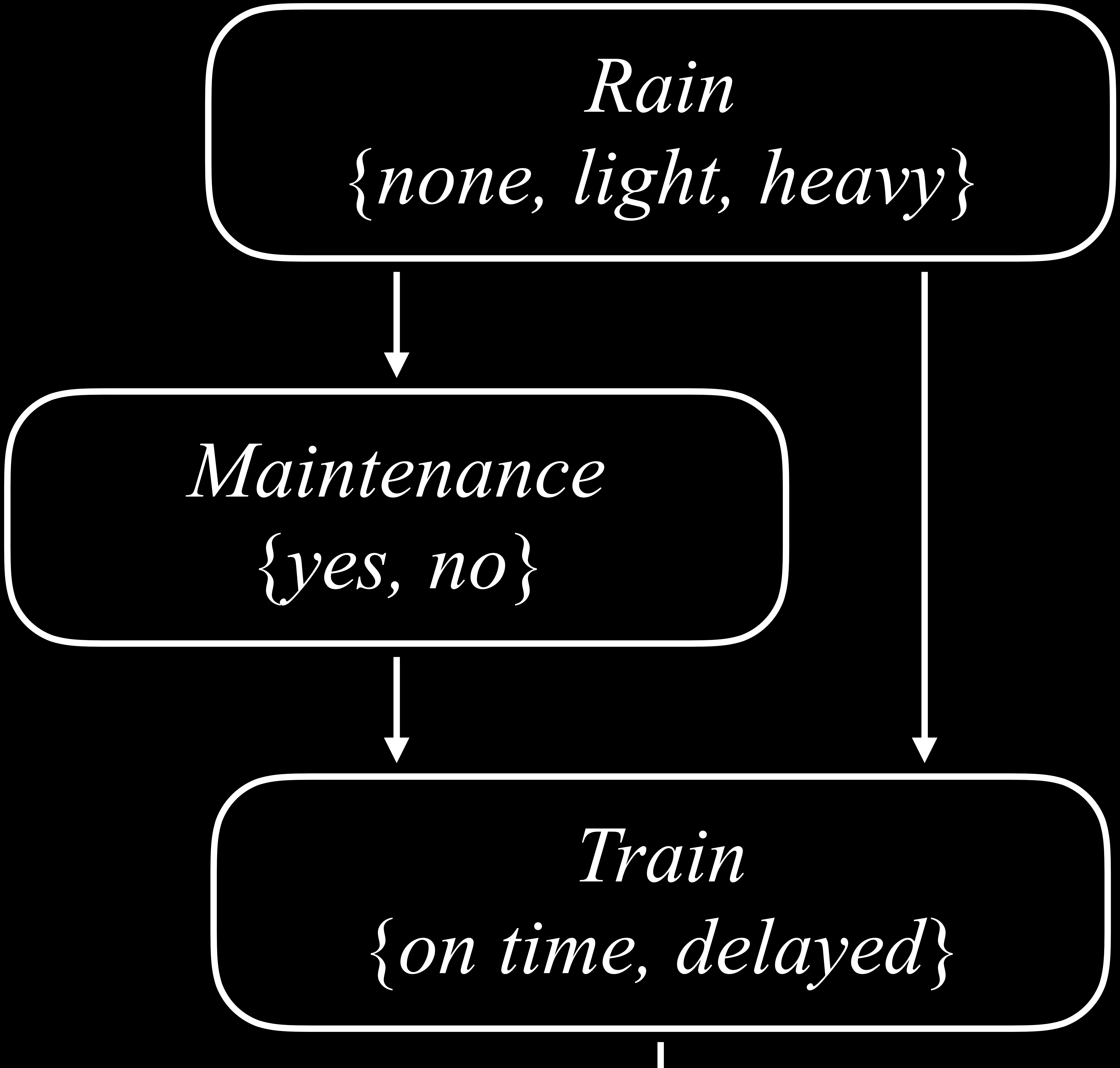
R = *light*
M = *yes*
T = *on time*
A = *attend*

T	*attend*	*miss*
on time	0.9	0.1
delayed	0.6	0.4



R = *light*	
M = *yes*	
T = *on time*	
A = *attend*	

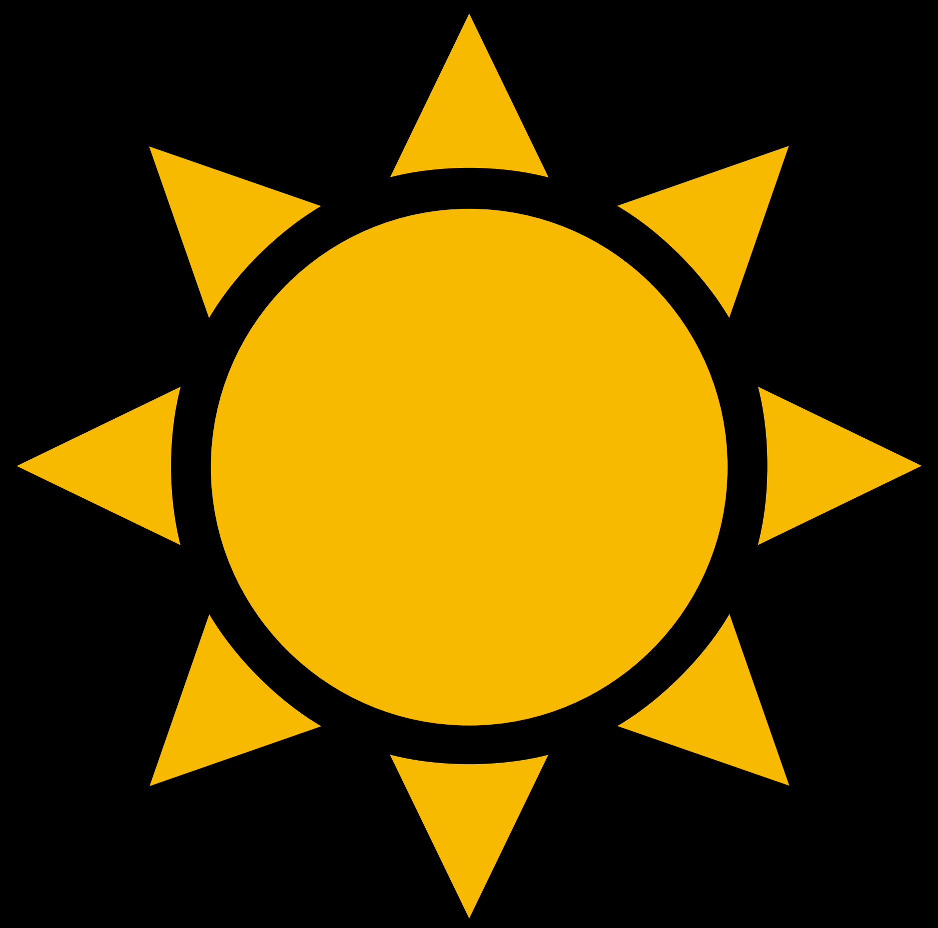
R	*M*	*on time*	*delayed*
none	*yes*	0.8	0.2
none	*no*	0.9	0.1
light	*yes*	0.6	0.4
light	*no*	0.7	0.3
heavy	*yes*	0.4	0.6
heavy	*no*	0.5	0.5



R = *light*	
M = *yes*	
T = *on time*	
A = *attend*	

R	***M***	*on time*	*delayed*
none	*yes*	0.8	0.2
none	*no*	0.9	0.1
light	*yes*	0.6	0.4
light	*no*	0.7	0.3
heavy	*yes*	0.4	0.6
heavy	*no*	0.5	0.5

Uncertainty over Time



X_t: Weather at time t

Markov assumption

the assumption that the current state depends on only a finite fixed number of previous states

Markov Chain

Markov chain

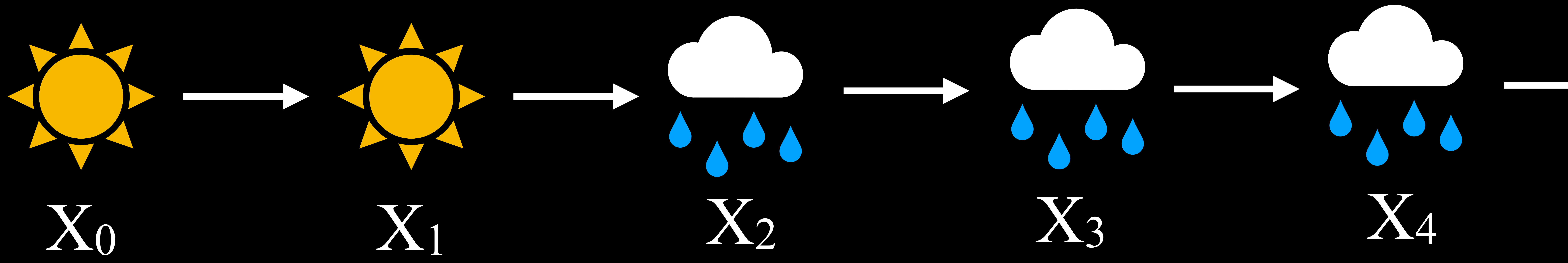
a sequence of random variables where the distribution of each variable follows the Markov assumption

Transition Model

Tomorrow (X_{t+1})

Today (X_t)

		
	0.8	0.2
	0.3	0.7


X_0
X_1
X_2
X_3
X_4

Sensor Models

Hidden State	Observation
robot's position	robot's sensor data
words spoken	audio waveforms
user engagement	website or app analytics
weather	umbrella

Hidden Markov Models

Hidden Markov Model

a Markov model for a system with hidden states that generate some observed event

Sensor Model

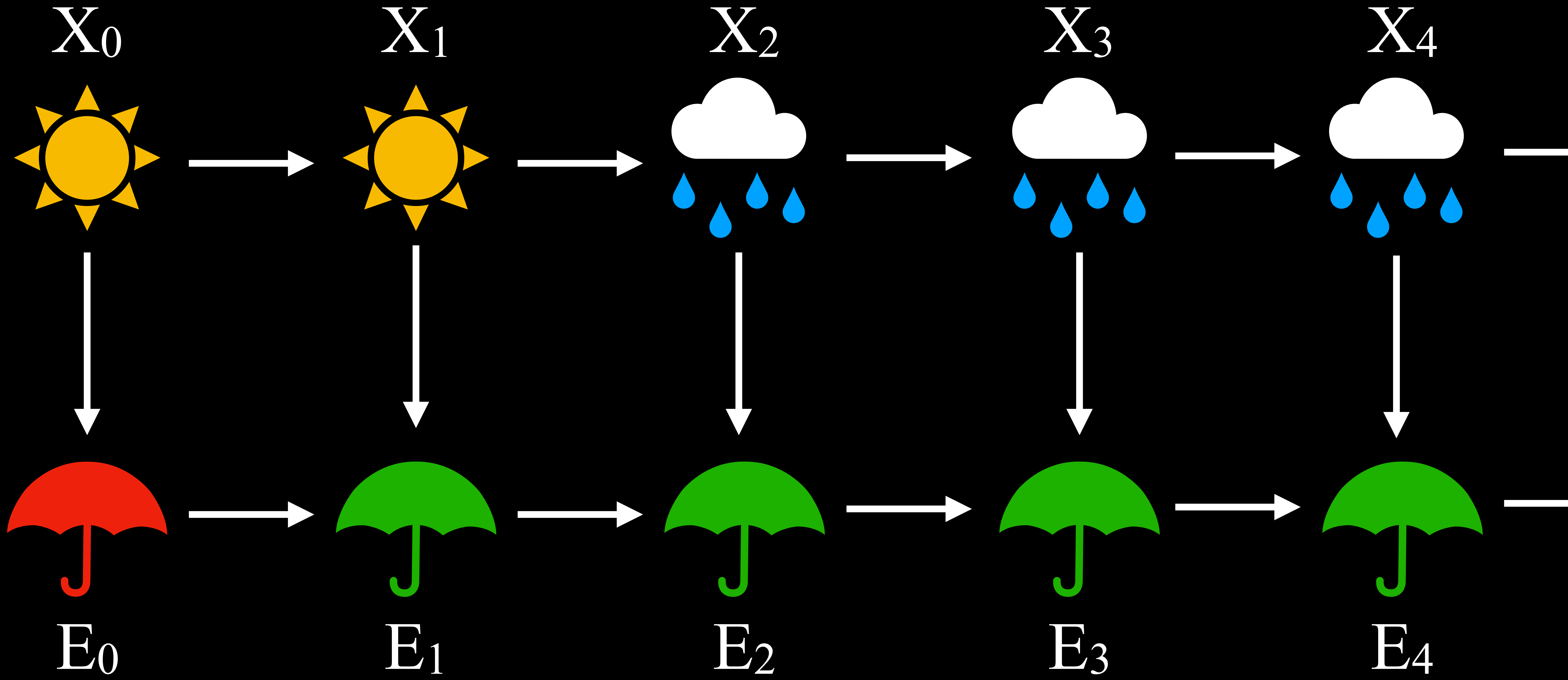
Observation (E_t)

State (X_t)

		
	0.2	0.8
	0.9	0.1

sensor Markov assumption

the assumption that the evidence variable depends only the corresponding state



X_0
X_1
X_2
X_3
X_4
E_0
E_1
E_2
E_3
E_4

Task	Definition
filtering	given observations from start until now, calculate distribution for **current** state
prediction	given observations from start until now, calculate distribution for a **future** state
smoothing	given observations from start until now, calculate distribution for **past** state
most likely explanation	given observations from start until now, calculate most likely **sequence** of states

Uncertainty

Introduction to

Artificial Intelligence

with Python